Strong Matching Preclusion under the Conditional Fault Model

Jung-Heum Park[a], Insung Ihm[b,*]

[a] *School of Computer Science and Information Engineering*
The Catholic University of Korea, Bucheon, Korea
[b] *Department of Computer Science and Engineering*
Sogang University, Seoul, Korea

Abstract

Strong matching preclusion that additionally permits more destructive vertex faults in a graph [J.-H. Park and I. Ihm, Strong matching preclusion, Theoretical Computer Science 412 (2011) 6409–6419] is an extended form of the original matching preclusion that assumes only edge faults [R.C. Brigham, F. Harary, E.C. Violin, and J. Yellen, Perfect-matching preclusion, Congressus Numerantium 174 (2005) 185–192]. In this paper, we study the problem of strong matching preclusion under the condition that no isolated vertex is created as a result of faults. After briefly discussing some fundamental classes of graphs in the point of the conditional matching preclusion, we establish the conditional strong matching preclusion number for the class of restricted hypercube-like graphs, which include most nonbipartite hypercube-like networks found in the literature.

Keywords: Perfect and almost perfect matching; Strong matching preclusion; Conditional vertex/edge faults; Restricted hypercube-like graph.

1. Introduction

1.1. Problem description

Given a graph $G = (V, E)$, a *matching* M of G is a set of pairwise nonadjacent edges. For G with an even number of vertices, a matching M that covers all vertices is called *perfect*. For G with an odd number of vertices, on the other hand, M is called *almost perfect* if it covers all but one vertex. A graph is *matchable* if it has either a perfect matching or an almost perfect matching. Otherwise, it is called *unmatchable*.

A *matching preclusion set* (MP set for short) of G is a set of edges whose deletion results in an unmatchable graph [3]. The *matching preclusion number* (MP number for short) of G, denoted by $mp(G)$, is defined as the minimum size of all possible MP sets of G. Any MP set of G whose size is $mp(G)$ is then regarded as a minimum MP set. As addressed in [3], the idea of matching preclusion offers a way of measuring the robustness of a given graph as a network topology with respect to link failures. That is, in the situation in which each node of a communication network is required to have a special neighboring partner node at any time, one that has a larger matching preclusion number may be considered as more robust in the event of possible link failures.

A trivial case of matching preclusion occurs when all edges in G incident to a single vertex (for G with an even number of vertices) or two particular vertices (for G with an odd number of vertices) are deleted, which models a situation where link failures are concentrated at only a very few nodes of a communication network. When such a case is unlikely to happen, a useful notion of matching preclusion is the *conditional matching preclusion*, which removes from consideration the matching preclusion set that produces a graph with an isolated vertex after edge deletion [4]. The *conditional matching preclusion number* (CMP number for short) of G, denoted by $mp_1(G)$, is

*Corresponding author
Email addresses: j.h.park@catholic.ac.kr (Jung-Heum Park), ihm@sogang.ac.kr (Insung Ihm)

then the minimum size of all such *conditional matching preclusion sets* (CMP sets for short) of G. Similarly to the unconditioned matching preclusion, a CMP set of G with $mp_1(G)$ elements is called a minimum CMP set of G.

Another type of failure in a communication network occurs through nodes, which is, in fact, more destructive. As an extended form of matching preclusion, the *strong matching preclusion* deals with the corresponding matching problem that additionally allows vertex deletion [12]. Then, finding a *strong matching preclusion set* (SMP set for short) of G means looking for a set of vertices and/or edges whose deletion leads to an unmatchable graph, where the notion of *strong matching preclusion number* (SMP number for short), denoted by $smp(G)$, and *minimum SMP set* are defined naturally. Note that the strong matching preclusion is more general than the problems discussed in [1, 8], which considered only vertex deletions.

For the same reason as in the original matching preclusion problem, a conditional version of strong matching preclusion, in which only the fault sets that do not create an isolated vertex are considered, also deserves investigations, complementing the four extended combinations of matching preclusion. In this article, we formulate the notion of *conditional strong matching preclusion*, and discuss its fundamental properties for some classes of graphs. Then, we focus on the restricted hypercube-like graphs, which represent most nonbipartite hypercube-like communication networks found in the literature, and study how robust they are to the conditional node and/or link failure. In particular, we rigorously derive their conditional strong matching preclusion number, and discuss some of the minimum sets that lead to it.

1.2. Previous Results

In their seminal paper, Brigham et al. established the matching preclusion numbers of four basic classes of graphs, namely, the Petersen graph, the complete graphs, the complete bipartite graphs, and the hypercubes, and classified their minimum matching preclusion sets [3]. Under the condition of no isolated vertex after deletion, Cheng et al. showed that the matching preclusion number generally increases for those graphs and also provided the minimal sets [4]. Since then, various classes of graphs for interconnection topology have been investigated to understand the robustness of the respective graph as an interconnection network without the no-isolated-vertex condition [7, 11, 15, 17] and with the condition [15, 5, 17]. Recently, Park and Ihm [12] studied how an additional permission of vertex deletion affects the robustness of graphs in terms of matching preclusion, and classified the minimum strong matching preclusion sets for a variety of graphs. Naturally, the strong matching preclusion numbers were found to be less than or equal to those of the original matching preclusion, although the number did not decrease for such graphs as restricted hypercube-like graphs and recursive circulants that have robust interconnections between nodes. Then, followed by this work, the strong matching preclusion problem was studied for some classes of graphs such as alternating group graphs and split-stars [2].

2. Preliminaries

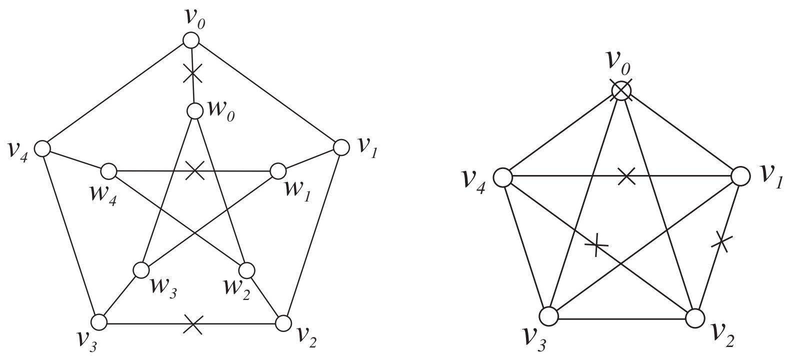
Given a graph G, it may happen that some of its vertices or edges become faulty. In this article, the fault vertex set and the fault edge set are denoted by F_v and F_e, respectively, which together form a *fault set* F of G ($F = F_v \cup F_e$). If a fault set F does not isolate a vertex, i.e. the graph $G \setminus F$ has no isolated vertex, it is said to be a *conditional fault set*. The main concern of this article is to study the conditional fault sets that preclude matchings in a given graph.

Definition 1. *A conditional fault set F of a graph G is called a* conditional strong matching preclusion set *(CSMP set for short) of G if $G \setminus F$ has neither a perfect matching nor an almost perfect matching. The minimum cardinality of all CSMP sets of G is denoted by $smp_1(G)$, and is said to be the* conditional strong matching preclusion number *(CSMP number for short) of G. A CSMP set whose size is $smp_1(G)$ is then called a* minimum CSMP set *of G.*

The number $smp_1(G)$ is naturally defined to be zero if G itself is unmatchable, whereas it is undefined if G has no CSMP set as in the case of the trivial graph with only one vertex.

From the fact that a matching preclusion set of a graph is a special strong matching preclusion set made of edges only, the following proposition is obvious.

Proposition 1. *For every graph G for which all the four numbers, $mp(G)$, $mp_1(G)$, $smp(G)$, and $smp_1(G)$ are well defined, $smp(G) \le smp_1(G) \le mp_1(G)$ and $smp(G) \le mp(G) \le mp_1(G)$.*



(a) The Petersen graph (b) The complete graph K_5

Figure 1: Two examples of the minimum conditional strong matching preclusion sets. Here, the symbol × marks fault elements.

Using the particular fact that $smp(G)$ is a lower bound of $smp_1(G)$, the CSMP numbers and sets of some graphs can simply be deduced from their SMP numbers and sets. For the Petersen graph G, for example, it was shown in Theorem 1 of [12] that $smp(G) = 3$ and each of its minimum SMP set isolates a vertex or is isomorphic to the form of $\{(v_0, w_0), (v_2, v_3), (w_1, w_4)\}$ (see Figure 1(a)). Trivially, the theorem has the following corollary for conditional strong matching preclusion.

Corollary 1. *For the Petersen graph G, $smp_1(G) = 3$. Furthermore, each of its minimum CSMP set is isomorphic to the form of $\{(v_0, w_0), (v_2, v_3), (w_1, w_4)\}$.*

Similarly, Theorem 2 and Theorem 3 of [12], which state the strong matching preclusion properties of complete graphs and connected regular bipartite graphs, respectively, also have the corresponding corollaries.

Corollary 2. *For a complete graph K_m with $m \geq 4$, $smp_1(K_m) = m - 1$. Furthermore, each of its minimum SMP set is $F_v \cup F_e$, where $|F_v| = m - 4$ and F_e forms a triangle in $K_m \setminus F_v$.*

Corollary 3. *For a connected m-regular bipartite graph G with $m \geq 3$, $smp_1(G) = 2$. Furthermore, each of its minimum CSMP set is a set of two vertices from the same partite set.*

Note that the CSMP number of K_m is well defined only when $m \geq 4$. See Figure 1(b) for an example of the minimum CSMP set of K_5.

Remark 1. It should be mentioned that there is no general order between $mp(G)$ and $smp_1(G)$. For example, for a 6-dimensional restricted hypercube-like graph G^6, $mp(G^6) = 6$ [11], whereas $smp_1(G^6) = 9$ as will be shown in this article. On the other hand, for the 3-dimensional hypercube Q_3, it is obvious that $mp(Q_3) = 3$ although $smp_1(Q_3) = 2$.

When isolated vertices are allowed after deletion, a simple way of precluding matchings in a graph G is to pick a vertex and then select a fault set F isolating the vertex so that the faulty graph $G \setminus F$ has an even number of vertices. Since, for an arbitrary vertex of degree at least one, there always exists an SMP set that isolates the vertex, it is clear that $smp(G) \leq \delta(G)$ for any graph G with no isolated vertices, where $\delta(G)$ is the minimum degree of G.

Under the condition of no isolated vertices, however, things become a little bit more complicated. Similar to the observation made in [4], an easy way to build a CSMP set is to try a fault set F that leaves, after deletion, a path (u, z, v) made of three vertices u, z, and v, where $d_{G\setminus F}(u)$ and $d_{G\setminus F}(v)$, the degrees of u and v in $G \setminus F$, respectively, are both one. If $G \setminus F$ has an even number of vertices, the resulting graph becomes unmatchable. Therefore, provided that the vertex function $N_G(\cdot)$ represents the set of all neighboring vertices in G, we can build a candidate CSMP set as follows (refer to Figure 2 to get an intuition).

Given a path (u, z, v) in a graph $G = (V, E)$, build a fault set, denoted by F_{uzv}, in such a way that

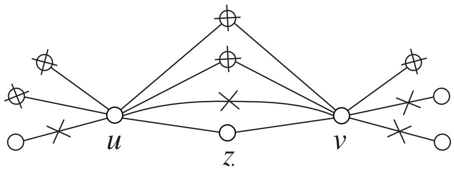


Figure 2: Construction of a simple conditional strong matching preclusion set F_{uzv}.

1. F_{uzv} contains every vertex $w \in (N_G(u) \cap N_G(v)) \setminus z$,
2. F_{uzv} contains the edge (u, v) if $(u, v) \in E$,
3. for every vertex $w \in N_G(u) \setminus N_G(v)$, F_{uzv} contains exactly one of w and (u, w), and
4. for every vertex $w \in N_G(v) \setminus N_G(u)$, F_{uzv} contains exactly one of w and (v, w).

Now, we have a fundamental proposition that will be frequently referred to in this article.

Proposition 2. *For an arbitrary path (u, z, v) in a graph G, F_{uzv} is a CSMP set of G if (i) there is no isolated vertex in $G \setminus F_{uzv}$, and (ii) $G \setminus F_{uzv}$ has an even number of vertices.*

Proof. Obvious. □

We call the CSMP set subject to Proposition 2 *trivial* as it is one of the simplest ways of building a CSMP set. Using the idea of the trivial CSMP set, we can easily establish an upper bound on its CSMP number.

Proposition 3. *If there exists a trivial CSMP set F_{uzv} for some path (u, z, v) in a graph G, then $smp_1(G) \leq d_G(u) + d_G(v) - 2 - g_G(u, v)$, where $g_G(u, v)$ is $|N_G(u) \cap N_G(v)|$ if $(u, v) \in E$, or $|N_G(u) \cap N_G(v)| - 1$ otherwise.*

Proof. The first two steps in the construction of F_{uzv} puts $g_G(u, v)$ elements into F_{uzv}. Then, the third and fourth steps adds $d_G(u) - 1 - g_G(u, v)$ and $d_G(v) - 1 - g_G(u, v)$ elements, respectively. Hence, the final F_{uzv} has $d_G(u) + d_G(v) - 2 - g_G(u, v)$ fault elements, proving the proposition. □

Finally, the parity of the lengths of paths spanning a given graph plays an important role in determining if the graph is matchable. In this article, a path in a graph is a sequence of distinct vertices such that consecutive ones are adjacent, and its *length* refers to the number of vertices in the sequence. A path is called an *even path* if its length is even. Otherwise, it is an *odd path*. Given these definitions, we have another straightforward but fundamental proposition.

Proposition 4. *Let F be a fault set of a graph G. Then, $G \setminus F$ is matchable if and only if $G \setminus F$ can be spanned by a set of disjoint even paths with at most one exceptional odd path.*

Proof. The necessity is obvious. An even path can be further partitioned into a set of paths of length two, i.e. matchings, while an odd one can be partitioned into matchings plus a single vertex. This implies that the sufficiency holds. □

3. Restricted Hypercube-like graphs

Let $\Phi(G_0, G_1)$ be a set of all bijections from $V(G_0)$ to $V(G_1)$ for two graphs G_0 and G_1, having the same number of vertices. Then, given a bijection $\phi \in \Phi(G_0, G_1)$, we denote by $G_0 \oplus_\phi G_1$ a new graph with vertex set $V(G_0) \cup V(G_1)$ and edge set $E(G_0) \cup E(G_1) \cup \{(v, \phi(v)) : v \in V(G_0)\}$. Here, G_0 and G_1 are called the *sides* of $G_0 \oplus_\phi G_1$, where every vertex v in one side has a unique neighbor $\bar{v}$ in the other one. Also, the edges $(v, \bar{v})$ connecting the two sides are called *cross edges*. To simplify the notation, we often omit the bijection ϕ from $\oplus_\phi$ when it is clear in the context.

Based on this graph constructor, Vaidya *et al.* [16] gave a recursive definition of a new type of graphs, called *hypercube-like graphs* (HL-graphs for short): $HL_0 = \{K_1\}$ and $HL_m = \{G_0 \oplus_\phi G_1 : G_0, G_1 \in HL_{m-1}, \phi \in \Phi(G_0, G_1)\}$ for $m \geq 1$. A graph in a subclass HL_m is made of 2^m vertices of degree m, and is called an *m-dimensional HL-graph*. Their network properties in the presence of faults have been studied in view of applications to parallel computing: hamiltonicity [13, 9], disjoint path covers [14], and diagnosability [10]. The HL-graphs have some simple but important structures, which will be used later.

Lemma 1. *Let G be an m-dimensional HL-graph with $m \geq 1$. Then, (a) $|N_G(u) \cap N_G(v)| \leq 2$ for every pair of vertices u and v in G, and (b) G contains no cycle of length three.*

Proof. These two facts can be verified easily by induction on m. Thus we omit the proof. □

A subset of the HL-graphs form an interesting group of graphs, called *restricted HL-graphs*, that have also been defined recursively by Park et al. [13]: $RHL_3 = HL_3 \setminus Q_3$ and $RHL_m = \{G_0 \oplus_\phi G_1 : G_0, G_1 \in RHL_{m-1}, \phi \in \Phi(G_0, G_1)\}$ for $m \geq 4$, where Q_3 indicates the 3-dimensional hypercube. Similar to the HL-graphs, a graph in RHL_m is called an *m-dimensional restricted HL-graph* and is denoted by G^m. The subclass RHL_3 contains only one graph, the recursive circulant $G(8,4)$ that is defined with vertex set $\{v_i : 0 \leq i \leq 7\}$ and edge set $\{(v_i, v_j) : j \equiv i+1 \text{ or } i+4 \pmod 8\}$ (see Figure 3 for a drawing of $G(8,4)$). Built from the nonbipartite graph $G(8,4)$, the restricted HL-graphs are nonbipartite, and, in fact, make up a proper subset of all nonbipartite HL-graphs. As addressed in [13], most nonbipartite hypercube-like network models found in the literature like crossed cube, Möbius cube, twisted cube, multiply twisted cube, Mcube, generalized twisted cube are, in fact, special restricted HL-graphs.

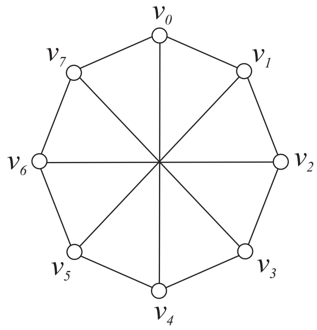


Figure 3: Recursive circulant $G(8,4)$.

To understand the robustness of the restricted HL-graphs in terms of matching, their MP, CMP, and SMP numbers have been analyzed in the previous works [11, 15, 12], which are summarized in Table 1. The multiple entries in the table indicate that there exist graphs corresponding to the respective matching preclusion numbers.

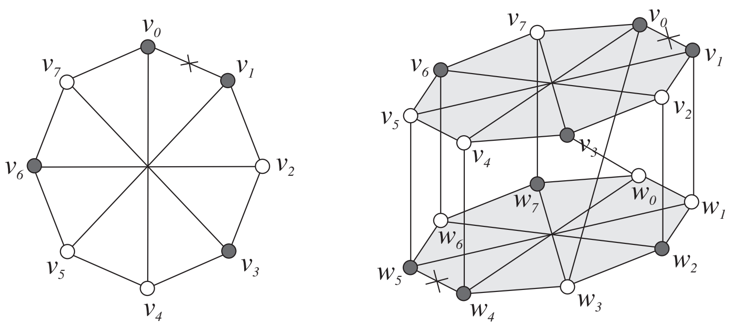
In this section, we complete the table by finding the CSMP numbers of the restricted HL-graphs. For this, we start with a lemma that describes an upper bound on $smp_1(G^m)$.

Lemma 2. *$smp_1(G^m) \leq 2m-3$ for every $m \geq 3$.*

Proof. To show the lemma holds, we prove by induction on m that there exists a fault set F_{uzv} of size $2m-3$ for some path (u, z, v) in G^m that satisfies the conditions of Proposition 2. When $m = 3$, it is easy to see that $(u, z, v) = (v_0, v_4, v_3)$ and $F_{uzv} = \{v_2, v_7, (v_0, v_1)\}$ satisfy the conditions, implying F_{uzv} is a trivial CSMP set of size three for $G^3 = G(8,4)$ (refer to Figure 3 again). When $m \geq 4$, let $G^m = G_0 \oplus G_1$ for some G_0 and G_1 in RHL_{m-1}. By the induction hypothesis, for some path (u, z, v) in G_0, there is a trivial CSMP set F^0_{uzv} of G_0 of size $2m-5$. Then, for $F_{uzv} = F^0_{uzv} \cup \{(u, \bar{u}), (v, \bar{v})\}$, it is straightforward to check that $G^m \setminus F_{uzv}$ has no isolated vertex and has an even number of vertices. Since the size of F_{uzv} is $2m-3$, the lemma is proven. □

Table 1: The four types of numbers for m-dimensional restricted HL-graphs G^m.

	$mp(G^m)$ [11]	$mp_1(G^m)$ [15]	$smp(G^m)$ [12]	$smp_1(G^m)$
$m = 3$	3	3	3	3
$m = 4$	4	5, 6	4	4, 5
$m = 5$	5	8	5	6, 7
$m \geq 6$	m	$2m-2$	m	$2m-3$



(a) G^3 w.r.t. (v_0, v_1)

(b) $H^4_{0,4}$

Figure 4: Two examples of white and black vertex sets.

Given this general upper bound, we try to see if it can be lowered further. First, we can immediately see from the following lemma proven in [12] that it cannot be for $m = 3$.

Lemma 3 ([12]). *$smp(G^3) = 3$ and each of its minimum SMP sets isolates a vertex or isomorphic to the form of* $\{(v_0, v_1), (v_0, v_4), (v_3, v_4)\}$, $\{(v_0, v_1), v_4, v_5\}$, $\{(v_0, v_1), v_2, v_4\}$, *or* $\{(v_0, v_1), v_2, v_7\}$.

Hence, we get the following corollary.

Corollary 4. *$smp_1(G^3) = 3$ and each of its minimum CSMP set is isomorphic to the form of* $\{(v_0, v_1), (v_0, v_4), (v_3, v_4)\}$, $\{(v_0, v_1), v_4, v_5\}$, $\{(v_0, v_1), v_2, v_4\}$, *or* $\{(v_0, v_1), v_2, v_7\}$.

Next, consider a 4-dimensional restricted HL-graph $G^4 = G_0 \oplus G_1$ with two sides G_0 and G_1 isomorphic to G^3. For the vertex sets $V(G_0) = \{v_0, v_1, \ldots, v_7\}$ and $V(G_1) = \{w_0, w_1, \ldots, w_7\}$, we assume that the two sides are respectively composed of *boundary edges* (v_i, v_{i+1}), (w_i, w_{i+1}) and *diagonal edges* (v_i, v_{i+4}), (w_i, w_{i+4}) for $0 \le i \le 7$ (from now on in this section, we assume all arithmetic on the indices of vertices is done modulo 8). Then, with respect to a boundary edge (v_i, v_{i+1}), we define the *white vertex* set $W_i = N_{H_i}(v_i) \cup N_{H_i}(v_{i+1})$ and the *black vertex* set $B_i = V(H_i) \setminus W_i$, where $H_i = G_0 \setminus (v_i, v_{i+1})$. As examplified in Figure 4(a), B_i forms an *independent set* in $G_0 \setminus (v_i, v_{i+1})$. That is, no two of the black vertices are adjacent in $G_0 \setminus (v_i, v_{i+1})$. Similarly, let W'_j and B'_j be the white and black vertex sets of G_1 with respect to a boundary edge (w_j, w_{j+1}).

For any two boundary edges (v_i, v_{i+1}) and (w_j, w_{j+1}), assume $H^4_{i,j}$ denotes a 4-dimensional restricted HL-graph, which is constructed from G_0 and G_1 in such a way that $\bar{W}_i = B'_j$ and $\bar{B}_i = W'_j$, where $\bar{X}$ denotes the neighbor set $\{\bar{x} : x \in X\}$ (see Figure 4(b) for an example of $H^4_{0,4}$). Then, the set of black vertices $B_i \cup B'_j$ forms an independent set in $H^4_{i,j} \setminus \{(v_i, v_{i+1}), (w_j, w_{j+1})\}$ again. Thus, a fault set F of size four containing (v_i, v_{i+1}) in G_0, (w_j, w_{j+1}) in G_1, and any two white vertices in $W_i \cup W'_j$ becomes a CSMP set of $H^4_{i,j}$ because there are 14 fault-free vertices and $|B_i \cup B'_j| = 8$, implying there exists no perfect matching in $H^4_{i,j} \setminus F$.

This fact has been discussed in [12] with respect to strong matching preclusion, and, in fact, these CSMP sets are the only possible CSMP sets of size four of the 4-dimensional restricted HL-graphs. We restate Theorem 5 of [12] here since it also gives important clues to the CSMP numbers of the restricted HL-graphs.

Theorem 1 ([12]). *For every $m \ge 3$, $smp(G^m) = m$. Furthermore, (a) for $m \ge 5$, each of its minimum SMP sets isolates a vertex, and (b) for $m = 4$, each of its minimum SMP sets of $G^4 = G_0 \oplus G_1$ either isolates a vertex or is a set composed of a boundary edge (v_i, v_{i+1}) of G_0, another boundary edge (w_j, w_{j+1}) of G_1, and two white vertices in $W_i \cup W'_j$ such that $\bar{W}_i = B'_j$ and $\bar{B}_i = W'_j$.*

Remark 2 ([12]). There exists a 4-dimensional restricted HL-graph such that every minimum SMP set of the graph is trivial. Let G be a graph in RHL_4 such that $\bar{v}_i = w_{3i}$ for every i. Then, for any set B_i of black vertices, $\bar{B}_i$ is consecutive, that is, $\bar{B}_i = \{w_j, w_{j+1}, w_{j+2}, w_{j+3}\}$ for some j. By Theorem 1, every minimum SMP set of G is trivial.

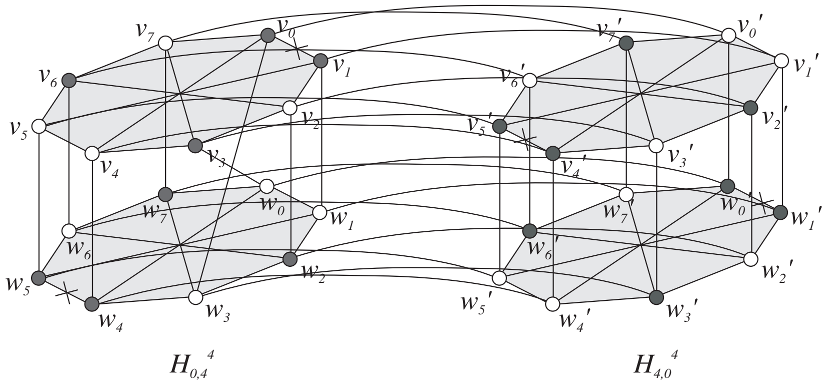


Figure 5: An illustration of $H^5_{0,4;4,0}$.

From this result, we are directly led to the next lemma.

Lemma 4. *(a) For any conditional fault set F of G^4 with $|F| = 4$, $G^4 \setminus F$ is unmatchable if and only if G^4 is isomorphic to $H^4_{i,j}$ for some i, j and $F = \{(v_i, v_{i+1}), (w_j, w_{j+1}), s, t\}$ for some $s, t \in W_i \cup W'_j$.*
(b) $smp_1(G^4) = 4$ or 5.

Proof. The statement (a) is immediate from Theorem 1(b). Furthermore, the theorem says that $smp(G^4) = 4$. Hence, by Proposition 1 and Lemma 2, $4 \le smp_1(G^4) \le 5$. By Remark 2, there exists a 4-dimensional restricted HL-graph that is not isomorphic to $H^4_{i,j}$, implying that there exists a 4-dimensional restricted HL-graph that should have a CSMP set of size five. □

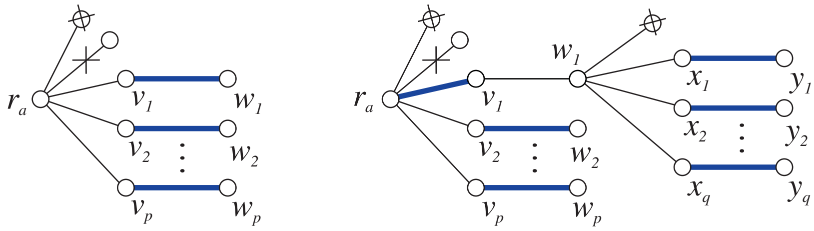
The graph corresponding to $H^4_{i,j}$ in RHL_5 can be constructed from the two 4-dimensional restricted HL-graphs with the CSMP number four. Consider two graphs $H^4_{i,j}$ and $H^4_{p,q}$, where $H^4_{i,j} = H_0 \oplus H_1$ and $H^4_{p,q} = H'_0 \oplus H'_1$ with $V(H_0) = \{v_0, v_1, \ldots, v_7\}$, $V(H_1) = \{w_0, w_1, \ldots, w_7\}$, $V(H'_0) = \{v'_0, v'_1, \ldots, v'_7\}$, and $V(H'_1) = \{w'_0, w'_1, \ldots, w'_7\}$. Similarly in the four dimensional case, assume that $W_{i,j}$, $B_{i,j}$ and $W'_{p,q}$, $B'_{p,q}$ are the sets of white and black vertices in $H^4_{i,j}$ and $H^4_{p,q}$, respectively.

Let $H^5_{i,j;p,q}$ denote a 5-dimensional restricted HL-graph $H^4_{i,j} \oplus H^4_{p,q}$ subject to $\bar{W}_{i,j} = B'_{p,q}$ and $\bar{B}_{i,j} = W'_{p,q}$. Then, the union of the black vertex sets, $B_{i,j} \cup B'_{p,q}$, becomes an independent set in $H^5_{i,j;p,q} \setminus \{(v_i, v_{i+1}), (w_j, w_{j+1}), (v'_p, v'_{p+1}), (w'_q, w'_{q+1})\}$ as before (see Figure 5). This implies that a fault set F of size six containing (v_i, v_{i+1}), (w_j, w_{j+1}), (v'_p, v'_{p+1}), (w'_q, w'_{q+1}), and any two white vertices in $W_{i,j} \cup W'_{p,q}$ forms a CSMP set of $H^5_{i,j;p,q}$ since $H^5_{i,j;p,q} \setminus F$ has 30 vertices and an independent set $B_{i,j} \cup B'_{p,q}$ of size 16.

Whereas $5 \le smp_1(G^5) \le 7$ due to Theorem 1, Proposition 1 and Lemma 2, Theorem 1(a) says that no CSMP set of size five is possible for 5-dimensional restricted HL-graphs, indicating that $smp_1(G^5)$ may be 6 or 7. So, $H^5_{i,j;p,q}$ is an example of G^5 with the minimum CSMP set of size six, which, in fact, is the only possible type of such sets of G^5 as will be discussed shortly. Now, with this background, we are ready to state the main theorem of this section that describes the CSMP numbers of G^m for $m \ge 5$. Here, a graph G is said to be *conditional f-fault matchable* if for any conditional fault set F with $|F| \le f$, $G \setminus F$ is matchable.

Theorem 2. *(a) Every m-dimensional restricted HL-graph G^m with $m \ge 6$ is conditional $(2m-4)$-fault matchable.*
(b) For any conditional fault set F of G^5 with $|F| \le 6$ $(= 2\cdot 5-4)$, $G^5 \setminus F$ is unmatchable if and only if G^5 is isomorphic to $H^5_{i,j;p,q}$ for some i, j, p, q and $F = \{(v_i, v_{i+1}), (w_j, w_{j+1}), (v'_p, v'_{p+1}), (w'_q, w'_{q+1}), s, t\}$ for some $s, t \in W_{i,j} \cup W'_{p,q}$.

Before presenting the proof of the theorem, we first derive two key lemmas (Lemmas 5 and 6) that give a useful insight into the properties of matchings in G^m. The first one is based on two fundamental facts on a class of bipartite graphs $G = C_n \oplus C_n$, constructed from two instances of the cycle graph C_n with n vertices (for readability of the proof of the lemma, we describe them in the appendix).



(a) M_0 with (v_i, w_i) all matched

(b) M'_0 with (x_j, y_j) all matched

Figure 6: Illustration of M_0 and M'_0.

Lemma 5. *Let F^4 and F^5 be any instances of the CSMP sets of size four and six of $H^4_{i,j}$ and $H^5_{i,j;p,q}$, whose structures are described in Lemma 4(a) and Theorem 2(b), respectively. Then, (a) $H^4_{i,j} \setminus (F^4 \cup \{x_1, x_2\})$ has a perfect matching for any pair of black vertices x_1 and x_2 in $B_i \cup B'_j$, and (b) $H^5_{i,j;p,q} \setminus (F^5 \cup \{x_1, x_2\})$ has a perfect matching for any pair of black vertices x_1 and x_2 in $B_{i,j} \cup B'_{p,q}$.*

Proof. Let G_0 and G_1 be the two sides of $H^4_{i,j}$, i.e. $H^4_{i,j} = G_0 \oplus G_1$, where G_i is isomorphic to $G(8,4)$, $i = 0, 1$. Then, $G_0 \setminus (v_i, v_{i+1})$ has an *alternating* hamiltonian cycle with index sequence $(i, i+4, i+3, i+2, i+1, i+5, i+6, i+7, i)$, in which no two consecutive vertices have the same color, and so does $G_1 \setminus (w_j, w_{j+1})$. Since $H^4_{i,j} \setminus \{(v_i, v_{i+1}), (w_j, w_{j+1})\}$ contains a bipartite graph in the form of $C_8 \oplus C_8$ as a spanning subgraph, the claim in (a) holds due to Lemma 7 in the appendix. Next, Lemma 8, also shown in the appendix, says that any bipartite graph in the form of $C_8 \oplus C_8$ has a hamiltonian cycle. This implies that for any $H^5_{i,j;p,q}$ with two sides $H^4_{i,j}$ and $H^4_{p,q}$, $H^5_{i,j;p,q} \setminus \{(v_i, v_{i+1}), (w_j, w_{j+1}), (v'_p, v'_{p+1}), (w'_q, w'_{q+1})\}$ contains a bipartite graph in the form of $C_{16} \oplus C_{16}$ as a spanning subgraph. Again, the claim in (b) follows immediately from Lemma 7. □

The next lemma suggests a direction to construct a perfect or almost perfect matching in a graph $G_0 \oplus G_1 \setminus F$. Let $F = F_0 \cup F_1 \cup F_{01}$, where F_0 and F_1 denote the sets of faulty vertices and/or edges in G_0 and G_1, respectively, and F_{01} represents the set of faulty edges, if any, joining vertices of the two sides G_0 and G_1. Then, we first find as large a matching in $G_0 \setminus F_0$ as possible and then, for every vertex r in G_0 that is *unmatched* by the matching, i.e. that is not covered by the matching, pair it with $\bar{r}$ in G_1. For this method to be viable, all such edges $(r, \bar{r})$ must be *free* with respect to F, i.e. $r, \bar{r}, (r, \bar{r}) \notin F$. The next lemma says that it is always possible.

Lemma 6. *Let F be a conditional fault set of an m-dimensional restricted HL-graph $G^m = G_0 \oplus G_1$ subject to $|F| \leq 2m - 4$. If $G_0 \setminus F_0$ has a matching M_0 with at least one unmatched vertex, then it has another matching M_1 with $|M_1| > |M_0|$ or has yet another matching M_2 with $|M_2| = |M_0|$ such that for every unmatched vertex r in G_0, the cross edge $(r, \bar{r})$ is free.*

Proof. Let $R = \{r_1, r_2, \ldots, r_k\}$ $(k \geq 1)$ be the set of all vertices in $G_0 \setminus F_0$ that are unmatched w.r.t. M_0. In this proof, we show that if we cannot find a matching M_1 with $|M_1| > |M_0|$, we can instead build a matching M_2 with a set of unmatched vertices $R' = R \cup \{r'_i\} \setminus r_i$ for some i such that $(r_i, \bar{r}_i)$ was not free but $(r'_i, \bar{r}'_i)$ is free. In the latter case, $|M_2| = |M_0|$ and the number of unmatched vertices of M_2 whose corresponding cross edges are not free decreases by one. It is clear that a repeated application of this process proves the lemma.

First, if $(r_i, \bar{r}_i)$ is free for every $r_i \in R$, the lemma is proven. Otherwise, let r_a be a vertex in R such that $(r_a, \bar{r}_a)$ is not free. Then, let $v_1, v_2, \cdots, v_p$ list all the vertices adjacent to r_a in G_0, for each of which the edge (r_a, v_i) is free in $G_0 \setminus F_0$ (see Figure 6(a)). There exists at least one such vertex, i.e. $p \geq 1$ since F_0, a subset of F, is a conditional fault set of G_0 (note that p is at most $m - 1$ since r_a has $m - 1$ incident edges in G_0). For each of the $m - 1 - p$ problematic edges, either the edge or the vertex other than r_a belongs to the fault set F_0. It is possible that both the edge and the vertex are contained in F_0. In that case, however, we only consider the fault vertex, and say that r_a is involved with $m - 1 - p$ fault elements in F_0.

Now, if some v_i is unmatched, we let $M_1 = M_0 \cup \{(r_a, v_i)\}$, completing the proof of the lemma. In the other case, where every v_i is matched, assume $(v_i, w_i) \in M_0$. If $(w_i, \bar{w}_i)$ is free for some w_i, we let $M_2 = M_0 \cup \{(r_a, v_i)\} \setminus (v_i, w_i)$, also completing the proof. The last case is when every v_i is matched and $(w_i, \bar{w}_i)$ is not free for all w_i. Consider a new matching $M_0' = M_0 \cup \{(r_a, v_1)\} \setminus (v_1, w_1)$ with the same number of matched edges as M_0, where the number of unmatched vertices whose cross edges are not free is also the same. Let $x_1, x_2, \ldots, x_q$ $(0 \le q \le m-2)$ be all the vertices in G_0 other than v_1 that are adjacent to w_1, for each of which the edge (w_1, x_j) is free (see Figure 6(b)). Similarly as before, w_1 is involved with $m-2-q$ fault elements in F_0.

Consider the first case in which there exists at least one such x_j $(q \ge 1)$. If there exists some x_j that is unmatched w.r.t. M_0', then we let $M_1 = M_0' \cup \{(w_1, x_j)\}$ and complete the proof. Otherwise, that is, if every x_j is matched, assume $(x_j, y_j) \in M_0'$. In this case, if there exists at least one y_j whose cross edge $(y_j, \bar{y}_j)$ is free, then we let $M_2 = M_0' \cup \{(w_1, x_j)\} \setminus (x_j, y_j)$, also completing the proof. If there exists no such y_j, i.e. $(y_j, \bar{y}_j)$ is not free for every y_j, we can see from all the assumptions made so far that the fault set F must contain at least the following many fault elements:

(i) 1 for the non-free $(r_a, \bar{r}_a)$,
(ii) $m-1-p$ for the problematic incident edges of r_a,
(iii) p for the non-free $(w_i, \bar{w}_i)$,
(iv) $m-2-q$ for the problematic incident edges of w_1,
(v) q for the non-free $(y_j, \bar{y}_j)$, and
(vi) -1 for the case where r_a and w_1 have a common neighbor other than v_1 (recall that Lemma 1 says there is at most one such common neighbor).

Then, the total number of fault elements amounts to $2m-3$, which is a contradiction to the assumption $|F| \le 2m-4$. Finally, if there exists no such x_j at all $(q = 0)$, we can finish the proof in the same way as the previous subcase. □

Finally, we are ready to give the proof of Theorem 2, which says that, for any conditional fault set F with $|F| \le 2m-4$, $G^m \setminus F$ is always matchable for all $m \ge 5$ except for the special case, where some F of size 6 $(= 2 \cdot 5 - 4)$ of some particular G^5 may make the graph unmatchable. For brevity of the proof, $I_G(v)$ denotes the set of all edges incident to vertex v in G, and a *p-matching* refers to a perfect or almost perfect matching. Throughout the proof, bear in mind that G^m is an m-regular graph with 2^m vertices.

Proof of Theorem 2. The proof is by induction on m. Let $G^m = G_0 \oplus G_1$ for some $G_0, G_1 \in RHL_{m-1}$. Consider an arbitrary conditional fault set F of G^m with $F = F_0 \cup F_1 \cup F_{01}$. It suffices to consider the case of $|F| = 2m-4$ because, if $|F| < 2m-4$, $G^m \setminus (F \cup F')$ will be shown to be matchable for some *virtual* fault set F' consisting of $2m-4-|F|$ edges that are fault-free with respect to F. In this proof, we assume without loss of generality that $|F_0| \ge |F_1|$. This implies that $|F_1| \le m-2$ and hence $G_1 \setminus F_1$ is always matchable since $smp(G^{m-1}) = m-1$. Remember that the fault set F_0 may not be conditional in G_0 if $|F| \ge m-1$, whereas F_1 is always conditional in G_1. The proof is divided into three cases.

Case 1: $|F_1| = m-2$ $(|F_0| = m-2$ and $|F_{01}| = 0)$. In this case, both $G_0 \setminus F_0$ and $G_1 \setminus F_1$ are matchable since, again, $smp(G^{m-1}) = m-1$. If at least one of $G_0 \setminus F_0$ and $G_1 \setminus F_1$ has a perfect matching, $G_0 \oplus G_1 \setminus F$ is matchable and we are done. Suppose that both $G_0 \setminus F_0$ and $G_1 \setminus F_1$ have only almost perfect matchings, indicating that both of them have odd number of vertices. For this case, let us consider the inductive step of $m \ge 6$ first. Observe that $F_0 \cup \{x\}$ may not be conditional in G_0, an $(m-1)$-regular graph, for at most one vertex x, and, similarly, $F_1 \cup \{y\}$ may not be conditional in G_1 for at most one vertex y. By avoiding the fault elements in F_0, F_1, and two possible such x and y, we can always find a free edge $(w, \bar{w})$ with $w \in V(G_0)$ such that $F_0 \cup \{w\}$ and $F_1 \cup \{\bar{w}\}$ are conditional in the respective sides. That is because there are 2^{m-1} candidate edges whereas at most $2m-2$ $(= (2m-4)+2)$ of them could be blocked, for which $2^{m-1} > 2m-2$ for any $m \ge 6$. Since $m-1$, the size of $F_0 \cup \{w\}$ and $F_1 \cup \{\bar{w}\}$, is always less than $2(m-1)-4$ for $m \ge 6$, $G_0 \setminus (F_0 \cup \{w\})$ and $G_1 \setminus (F_1 \cup \{\bar{w}\})$ have perfect matchings, say, M_0 and M_1, respectively, by the induction hypothesis. Hence, $G_0 \oplus G_1 \setminus F$ is matchable by $M_0 \cup M_1 \cup \{(w, \bar{w})\}$.

Now, take a look at the basis step of $m = 5$. We first consider the very special case, in which G_0 and G_1 are respectively isomorphic to $H^4_{i,j}$ and $H^4_{p,q}$ for some i, j, p, q, $F_0 = \{(v_i, v_{i+1}), (w_j, w_{j+1}), s\}$ for some white vertex $s \in W_{i,j}$, and $F_1 = \{(v'_p, v'_{p+1}), (w'_q, w'_{q+1}), t\}$ for some white vertex $t \in W'_{p,q}$. If there exists an edge $(w, \bar{w})$ with $w \in V(G_0)$ such that w and $\bar{w}$ are both black, then $(w, \bar{w})$ must be free and, by Lemma 4(a), $G_0 \setminus (F_0 \cup \{w\})$ and $G_1 \setminus (F_1 \cup \{\bar{w}\})$

are both matchable and thus have perfect matchings M_0 and M_1, respectively. Then, $G_0 \oplus G_1 \setminus F$ is matchable by $M_0 \cup M_1 \cup \{(w, \bar{w})\}$. If no such edge exists, every cross edge between G_0 and G_1 joins a pair of black and white vertices, implying that $G_0 \oplus G_1$ is isomorphic to $H^5_{i,j;p,q}$. As we discussed earlier, $G_0 \oplus G_1 \setminus F$ is not matchable since it has an independent set $B_{i,j} \cup B'_{p,q}$ of size 16.

Next, for the remaining case, assume w.l.o.g. that either G_0 is not isomorphic to $H^4_{i,j}$ for any i, j, or G_0 is isomorphic to $H^4_{i,j}$ for some i, j but F_0 is not in the form of $\{(v_i, v_{i+1}), (w_j, w_{j+1}), s\}$ for $s \in W_{i,j}$. We claim that there always exists a free edge $(w, \bar{w})$ with $w \in V(G_0)$ such that $G_0 \setminus (F_0 \cup \{w\})$ and $G_1 \setminus (F_1 \cup \{\bar{w}\})$ are both matchable by perfect matchings, which will complete the proof of this case. Recall that Lemma 4(a) implies that $G_0 \setminus (F_0 \cup \{x\})$ is matchable for any vertex x in $G_0 \setminus F_0$ provided that $F_0 \cup \{x\}$ is conditional in G_0. To assure that no isolated vertex is created by $F_0 \cup \{w\}$, we may have to avoid at most 4 vertices (at most three in F_0 and at most one outside F_0) in choosing w in G_0. Also, if G_1 is isomorphic to $H^4_{p,q}$ for some p, q and $F_1 = \{(v'_p, v'_{p+1}), (w'_q, w'_{q+1}), t\}$ for some $t \in W'_{p,q}$, $G_1 \setminus (F_1 \cup \{y\})$ is matchable for any black vertex y in G_1, and thus it is sufficient to avoid the 8 white vertices in selecting $\bar{w}$ in G_1. For other G_1 and/or F_1, $G_1 \setminus (F_1 \cup \{y\})$ is matchable for any vertex y in $G_1 \setminus F_1$ provided that $F_1 \cup \{y\}$ is conditional, again, suggesting that at most 4 vertices in G_1 must be avoided in the choice of $\bar{w}$. Therefore, at most $4 + \max\{8, 4\}$ cross edges may cause a trouble but we have 16 candidates. So, the claim holds.

Case 2: $|F_1| \leq m - 3$ and $|F_0| < |F|$ ($|F_0| \leq 2m - 5$ and $|F_1| + |F_{01}| \geq 1$). Suppose that there are two isolated vertices z_1 and z_2 in $G_0 \setminus F_0$. If $(z_1, z_2) \in E(G_0)$, then $|F_0|$ must be at least $2(m - 1) - 1$ because there is no cycle of length three by Lemma 1. If not, it must be at least $2(m - 1) - 2$ because there are at most two common neighbors of z_1 and z_2 according to the same lemma. In either case, we are led to a contradiction to the assumption that $|F_0| < |F| = 2m - 4$. Thus, $G_0 \setminus F_0$ may have at most one isolated vertex. Also, a similar counting may prove that there is at most one fault vertex v in F_0 such that v is isolated in $G_0 \setminus (F_0 \setminus v)$. Furthermore, it is trivial to see that such isolated vertices, whether they are in F_0 or not, may not coexist. Now, the remaining proof proceeds by considering three simple subcases first, and then handling the remaining, more general one.

The first subcase is that there is a vertex z in $G_0 \setminus F_0$ such that $I_{G_0}(z) \subseteq F_0$, in which case $(z, \bar{z})$ must be free since F is conditional. If we let $F'_0 = (F_0 \setminus I_{G_0}(z)) \cup \{z\}$, then $|F'_0| = |F_0| - (m - 1) + 1 \leq m - 3$, and $G_0 \setminus F'_0$ is matchable due to the fact that $smp(G^{m-1}) = m - 1$. If $G_0 \setminus F'_0$ has a perfect matching, say, M_0, we have a desired p-matching $M_0 \cup M_1 \cup \{(z, \bar{z})\}$ for a p-matching M_1 in $G_1 \setminus (F_1 \cup \{\bar{z}\})$, which must exist as $|F_1| + 1 \leq m - 2$. If $G_0 \setminus F'_0$ has only an almost perfect matching, we will claim that there exists a free edge $(w, \bar{w})$ with $w \in V(G_0)$ such that $G_0 \setminus (F'_0 \cup \{w\})$ has a perfect matching M_0 and $G_1 \setminus (F_1 \cup \{\bar{z}, \bar{w}\})$ has a p-matching M_1, which indicates the existence of a desired matching $M_0 \cup M_1 \cup \{(z, \bar{z}), (w, \bar{w})\}$. For the proof of the claim, note that $G_0 \setminus (F'_0 \cup \{x\})$ is matchable for any vertex x in $G_0 \setminus F'_0$ since $|F'_0| + 1 \leq m - 2$. If it happens that G_1 is isomorphic to $H^4_{p,q}$ for some p, q, $F_1 = \{(v'_p, v'_{p+1}), (w'_q, w'_{q+1})\}$, and $\bar{z} \in W'_{p,q}$, then $G_1 \setminus (F_1 \cup \{\bar{z}, y\})$ is matchable only for a black vertex y in G_1 (there are 8 troublesome white vertices in G_1); otherwise, $G_1 \setminus (F_1 \cup \{\bar{z}, y\})$ is matchable for any vertex y in $G_1 \setminus (F_1 \cup \{\bar{z}\})$ provided that $F_1 \cup \{\bar{z}, y\}$ is conditional. Since it suffices to avoid at most $(2m - 4) + 1 + 8$ cross edges among 2^{m-1} of them, we can always find such a free edge $(w, \bar{w})$.

The second subcase is that F_0 contains a fault vertex v with $I_{G_0}(v) \subseteq F_0$. Then, given a new fault set $F' = F \setminus I_{G_0}(v)$ with $m - 3$ ($= (2m - 4) - (m - 1)$) fault elements, $G \setminus F$ is identical to $G \setminus F'$. Since $smp(G^m) = m$, $G \setminus F'$ and hence $G \setminus F$ is matchable.

The third subcase is that F_0 contains only edges, in which we may safely assume that F_0, having at most $2m - 5$ fault edges, is conditional in G_0 because the non-conditional case was covered by the first subcase. If $m \geq 6$, or $m = 5$ and $|F_0| \leq 4$, $G_0 \setminus F_0$ has a perfect matching M_0 since $mp_1(G^{m-1}) = 2m - 4$ for any $m \geq 6$ and $mp_1(G^4) \geq 5$ (refer to Table 1 again). Thus, we have a desired matching $M_0 \cup M_1$ for a p-matching M_1 in $G_1 \setminus F_1$. If $m = 5$ and $|F_0| = 5$, it is clear that there is a fault edge (x, y) in F_0 such that $(x, \bar{x})$ and $(y, \bar{y})$ are both free. Let M_0 be a perfect matching in $G_0 \setminus (F_0 \setminus (x, y))$, which must exist since $|F_0 \setminus (x, y)| = 4 < mp_1(G_0)$. If $(x, y) \notin M_0$, $M_0 \cup M_1$ is a desired matching for a p-matching M_1 in $G_1 \setminus F_1$. Otherwise, $(M_0 \setminus (x, y)) \cup M_1 \cup \{(x, \bar{x}), (y, \bar{y})\}$ becomes a desired matching for a p-matching M_1 in $G_1 \setminus (F_1 \cup \{\bar{x}, \bar{y}\})$.

Now, hereafter in the proof of ***Case 2***, we consider the remaining subcase, where F_0 contains at least one fault vertex, and there is no vertex, whether it is in F_0 or not, all of whose incident edges belong to F_0. For the convenience of the proof, we define a fault set F_0' as follows.

$$F_0' = \begin{cases} F_0 & \text{if } F_0 \text{ is conditional in } G_0 \text{ and } |F_0| \le 2m-6, \\ F_0 \setminus v_f & \text{otherwise,} \end{cases}$$

where v_f is a fault vertex in F_0 adjacent to a vertex z via a fault-free edge if either $z \notin F_0$ is isolated in $G_0 \setminus F_0$, or $z \in F_0$ is isolated in $G_0 \setminus (F_0 \setminus z)$; otherwise, an arbitrary fault vertex in F_0 is chosen as v_f. Then, F_0' becomes a conditional fault set in G_0 with at most $2m-6$ fault elements, i.e. $|F_0'| \le 2m-6$. There are two cases.

Case 2.1: $G_0 \setminus F_0'$ is matchable. When $G_0 \setminus F_0'$ has a perfect matching, there are two possibilities: (i) $G_0 \setminus F_0$ has a perfect matching ($F_0' = F_0$), or (ii) has an almost perfect matching ($F_0' = F_0 \setminus v_f$). When $G_0 \setminus F_0'$ has an almost perfect matching M_0', there are three possibilities: (iii) $G_0 \setminus F_0$ has a perfect matching ($F_0' = F_0 \setminus v_f$ and v_f is not matched by M_0'), (iv) has an almost perfect matching ($F_0' = F_0$), or (v) has a matching with two unmatched vertices ($F_0' = F_0 \setminus v_f$ and v_f is matched by M_0').

Through this observation, we are led to three cases in terms of matchings in $G_0 \setminus F_0$. First, if $G_0 \setminus F_0$ has a perfect matching M_0 (cases (i) and (iii)), we have a desired matching $M_0 \cup M_1$ for a p-matching M_1 in $G_1 \setminus F_1$. Second, suppose that $G_0 \setminus F_0$ has an almost perfect matching (cases (ii) and (iv)). Then, Lemma 6 implies that there is an almost perfect matching M_0 in $G_0 \setminus F_0$ with one unmatched vertex r_1 such that $(r_1, \bar{r}_1)$ is free. So, we also have a desired matching $M_0 \cup M_1 \cup \{(r_1, \bar{r}_1)\}$ for a p-matching M_1 in $G_1 \setminus (F_1 \cup \{\bar{r}_1\})$.

Third, let $G_0 \setminus F_0$ have a matching M_0 with two unmatched vertices r_1 and r_2 (case (v)), in which $(r_1, \bar{r}_1)$ and $(r_2, \bar{r}_2)$ are both free, again, by Lemma 6. If $|F_1| \le m-4$, $G_1 \setminus (F_1 \cup \{\bar{r}_1, \bar{r}_2\})$ has a p-matching M_1 and thus we get a desired matching $M_0 \cup M_1 \cup \{(r_1, \bar{r}_1), (r_2, \bar{r}_2)\}$. On the other hand, if $|F_1| = m-3$, it follows that $|F_0| \le m-1$. From the definition of F_0' and the fact that $F_0' = F_0 \setminus v_f$ and $|F_0| \le m-1 \le 2m-6$ for $m \ge 5$, we can see that F_0 must not be conditional in G_0. Thus, there exists an isolated vertex z in $G_0 \setminus F_0$ and it must be that $|F_0| = m-1$.

For this last case of $|F_0| = m-1$ and $|F_1| = m-3$, in which $G_0 \setminus F_0$ has an isolated vertex z, we claim that there exists another free edge $(w, \bar{w})$, other than $(z, \bar{z})$, with $w \in V(G_0)$ such that $G_1 \setminus (F_1 \cup \{\bar{z}, \bar{w}\})$ is matchable. Since the fault set $F_1 \cup \{\bar{z}, \bar{w}\}$ has $m-1$ elements, and $m-1 < 2(m-1)-4$ for all $m \ge 6$, it is sufficient by the induction hypothesis to choose w such that $F_1 \cup \{\bar{z}, \bar{w}\}$ is conditional in G_1, which is clearly possible. The case of $m = 5$, however, needs a special treatment. If it happens to be that G_1 is isomorphic to $H^4_{p,q}$ for some p, q, $F_1 = \{(v_p', v_{p+1}'), (w_q', w_{q+1}')\}$, and $\bar{z} \in W_{p,q}'$, then $G_1 \setminus (F_1 \cup \{\bar{z}, y\})$ is matchable only for a black vertex y in G_1; otherwise, $G_1 \setminus (F_1 \cup \{\bar{z}, y\})$ is matchable for any vertex y in $G_0 \setminus (F_1 \cup \{\bar{z}\})$ provided that $F_1 \cup \{\bar{z}, y\}$ is conditional. A similar counting argument as before ($(2m-4)+1+8 < 2^{m-1}$) implies that such a choice of $(w, \bar{w})$ is possible.

Now, let $F_0'' = F_0' \cup \{w\} = (F_0 \setminus v_f) \cup \{w\}$. Then, $|F_0''| = m-1$ and F_0'' is now conditional in G_0. Furthermore, even if G_0 is isomorphic to $H^4_{i,j}$ for some i, j, F_0'' cannot be in the form of $\{(v_i, v_{i+1}), (w_j, w_{j+1}), s, t\}$ for any $s, t \in W_{i,j}$. Thus, $G_0 \setminus F_0''$ is matchable, implying that $G_0 \setminus F_0''$ has a perfect matching M_0 since $G_0 \setminus F_0$ has a matching with two unmatched vertices and has an even number of vertices. As $(z, v_f) \in M_0$, we have a desired matching $(M_0 \setminus (z, v_f)) \cup M_1 \cup \{(z, \bar{z}), (w, \bar{w})\}$ for some p-matching M_1 in $G_1 \setminus (F_1 \cup \{\bar{z}, \bar{w}\})$.

Case 2.2: $G_0 \setminus F_0'$ is not matchable. Recall that F_0' has been defined to be conditional in G_0 and $|F_0'| \le 2m-6$. For $G_0 \setminus F_0'$ to be unmatchable, we can have the following two cases only, which implies that $|F_0'| = 2m-6$, and hence $|F_0| = 2m-6$ or $2m-5$ depending on whether $F_0' = F_0$ or $F_0 \setminus v_f$:

(i) $m = 5$ and G_0 is isomorphic to $H^4_{i,j}$ for some i, j and $F_0' = \{(v_i, v_{i+1}), (w_j, w_{j+1}), s, t\}$ for some $s, t \in W_i \cup W_j'$ (by Lemma 4(a)), or

(ii) $m = 6$ and G_0 is isomorphic to $H^5_{i,j;p,q}$ for some i, j, p, q and $F_0' = \{(v_i, v_{i+1}), (w_j, w_{j+1}), (v_p', v_{p+1}'), (w_q', w_{q+1}'), s, t\}$ for some $s, t \in W_{i,j} \cup W_{p,q}'$ (by the induction hypothesis).

Consider the first case of $|F_0| = 2m-6$, where $F_0' = F_0$ and $|F_1 \cup F_{01}| = 2$. For $m = 5$, if $G_0 \oplus G_1$ is isomorphic to $H^5_{i,j;p,q}$ for some i, j, p, q, $F_0 = \{(v_i, v_{i+1}), (w_j, w_{j+1}), s, t\}$ for some $s, t \in W_{i,j}$, and $F_1 = \{(v_p', v_{p+1}'), (w_q', w_{q+1}')\}$, then $G_0 \oplus G_1 \setminus F$ is not matchable as discussed earlier. Suppose otherwise, for which we claim that there exist two free edges $(w_1, \bar{w}_1)$, $(w_2, \bar{w}_2)$ with black vertices $w_1, w_2 \in V(G_0)$ such that $G_1 \setminus (F_1 \cup \{\bar{w}_1, \bar{w}_2\})$ is matchable: if G_1 is isomorphic to $H^4_{p,q}$ for some p, q and $F_1 = \{(v_p', v_{p+1}'), (w_q', w_{q+1}')\}$, there must exist at least one free edge $(w_1, \bar{w}_1)$ joining a pair of black vertices. Then, it is enough to select another free edge $(w_2, \bar{w}_2)$ with black vertex w_2 since $F_1 \cup \{\bar{w}_1, \bar{w}_2\}$

becomes conditional in G_1 and cannot make G_1 unmatchable. If G_1 and/or F_1 are supposed otherwise, it is possible to pick up two free edges $(w_1, \bar{w}_1)$ and $(w_2, \bar{w}_2)$ such that w_1 and w_2 are black and $F_1 \cup \{\bar{w}_1, \bar{w}_2\}$ is conditional in G_1 since $2^{5-2} - |F_1 \cup F_{01}| - 1 \geq 2$. Then, by Lemma 4(a), $G_1 \setminus (F_1 \cup \{\bar{w}_1, \bar{w}_2\})$ is matchable. Once the claim is proved, we have a desired matching $M_0 \cup M_1 \cup \{(w_1, \bar{w}_1), (w_2, \bar{w}_2)\}$, where M_0 is a perfect matching in $G_0 \setminus (F_0 \cup \{w_1, w_2\})$, which exists by Lemma 5, and M_1 is a p-matching in $G_1 \setminus (F_1 \cup \{\bar{w}_1, \bar{w}_2\})$. The proof for $m = 5$ suggests an easier proof for $m = 6$: $G_1 \setminus (F_1 \cup \{x, y\})$ is matchable for any pair of vertices x and y in $G_1 \setminus F_1$ since $|F_1| + 2 < 6 - 1$. So, we can always choose two free edges $(w_1, \bar{w}_1)$ and $(w_2, \bar{w}_2)$ as mentioned above since $2^{6-2} - |F_1 \cup F_{01}| \geq 2$.

Now, consider the second case of $|F_0| = 2m - 5$, where $F'_0 = F_0 \setminus v_f$, $|F_1 \cup F_{01}| = 1$, and F_0 contains two white fault vertices s and t. If v_f is a black vertex, $F'_0 = F_0 \setminus s$, in which s replaces v_f, is another valid definition of F'_0. The new F'_0, containing a black vertex, is also conditional in G_0. Hence, $G_0 \setminus F'_0$ is matchable for $m = 5$ and 6 by Lemma 4(a) and the induction hypothesis, which leads to the subcase ***Case 2.1***. Hereafter, we assume that $F'_0 = F_0 \setminus v_f$ for some white vertex v_f. When $m = 6$, we select two free edges $(w_1, \bar{w}_1)$ and $(w_2, \bar{w}_2)$ with two black vertices w_1 and $w_2 \in V(G_0)$, which always exist. Then, $G_0 \setminus (F'_0 \cup \{w_1, w_2\})$ has a perfect matching as before, indicating that $G_0 \setminus (F_0 \cup \{w_1, w_2\})$ has an almost perfect matching M_0 with an unmatched vertex y. According to Lemma 6, we can say that $(y, \bar{y})$ is free. Also, $G_1 \setminus (F_1 \cup \{\bar{w}_1, \bar{w}_2, \bar{y}\})$ has a p-matching M_1 since $|F_1| + 3 < 6 - 1$. Thus, we have a desired matching $M_0 \cup M_1 \cup \{(w_1, \bar{w}_1), (w_2, \bar{w}_2), (y, \bar{y})\}$.

The proof for $m = 5$ is the same except that we must carefully select the free edges so that $G_1 \setminus (F_1 \cup \{\bar{w}_1, \bar{w}_2, \bar{y}\})$ is guaranteed to be matchable. If there are two free edges $(w_1, \bar{w}_1)$ and $(w_2, \bar{w}_2)$ such that $(\bar{w}_1, \bar{w}_2)$ is an edge of G_1, whether it is faulty or not, then $F_1 \cup \{\bar{w}_1, \bar{w}_2, \bar{y}\}$ will be conditional for any y by Lemma 1(b), and thus $G_1 \setminus (F_1 \cup \{\bar{w}_1, \bar{w}_2, \bar{y}\})$ is matchable by Lemma 4(a). It remains to prove the existence of such free edges. First of all, there are at least seven free edges $(w_i, \bar{w}_i)$ with w_i being a black vertex in G_0. However, note that the size of an independent set in G_1, which is isomorphic to $G(8, 4) \oplus G(8, 4)$, is at most six since that in $G(8, 4)$ is at most three. Therefore, there must exist a pair of free edges $(w_1, \bar{w}_1)$ and $(w_2, \bar{w}_2)$ among the seven such that $(\bar{w}_1, \bar{w}_2)$ is an edge of G_1.

Case 3: $|F_0| = |F| = 2m - 4$ ($|F_1| = |F_{01}| = 0$). In this last case, we first claim that $G_0 \setminus F_0$ may have at most two isolated vertices. Suppose for a contradiction that there are three isolated vertices z_1, z_2, and z_3 in $G_0 \setminus F_0$, for which $|N_{G_0}(z_i) \cap N_{G_0}(z_j)| \leq 2$ for all $i \neq j$ by Lemma 1(a). If $|N_{G_0}(z_i) \cap N_{G_0}(z_j)| < 2$ for some $i \neq j$, the number of fault elements involved in isolating z_i and z_j must be at least $2(m - 1) - 1 = 2m - 3$, which contradicts the assumption of $|F| = 2m - 4$. Thus, it must be that $|N_{G_0}(z_i) \cap N_{G_0}(z_j)| = 2$ for all $i \neq j$, but $(z_i, z_j) \notin E(G_0)$ by Lemma 1(b). In this case, if we define S to be $N_{G_0}(z_1) \cap N_{G_0}(z_2) \cap N_{G_0}(z_3)$, then $|S| \leq 2$. If $S = \{x, y\}$ for some x, y, then $\{z_1, z_2, z_3\} \subseteq N_{G_0}(x) \cap N_{G_0}(y)$, which contradicts Lemma 1(a). If $S = \{x\}$ for some x, then the number of fault elements, isolating the three z_i's, is at least $3(m - 4) + 3 + 1 = 3m - 8$, which is more than $|F|$ for $m \geq 5$. If $S = \emptyset$ and $m \geq 6$, then the number of fault elements that isolate the three z_i's is at least $3(m - 5) + 2 \cdot 3 = 3m - 9$, which is also greater than $|F|$. Finaly, when $S = \emptyset$ and $m = 5$, let $G_0 = G_0^0 \oplus G_0^1$, where the two sides G_0^0 and G_0^1 are isomorphic to $G(8, 4)$. If two different z_i's are contained in one side, their two common neighbors must also be in the same side due to the definition of the graph. Hence, it is impossible for all the three z_i's to be located in the same side of G^0, or the three of them with their six distinct common neighbors would sum up to more than eight vertices of $G(8, 4)$. Thus, assume w.l.o.g. that $z_1, z_2 \in V(G_0^0)$ and $z_3 \in V(G_0^1)$. Then, the condition of $|N_{G_0}(z_1) \cap N_{G_0}(z_3)| = 2$ implies that the vertex $x \in V(G_0^0)$ adjacent to z_3 via a cross edge must be in $N_{G_0}(z_1) \cap N_{G_0}(z_3)$, and similarly, in $N_{G_0}(z_2) \cap N_{G_0}(z_3)$, which contradicts the assumption of $S = \emptyset$. This ends the proof of our claim.

In addition to the claim, it can be shown, similarly to the previous case, that there may exist at most one fault vertex v in F_0 such that v is isolated in $G_0 \setminus (F_0 \setminus v)$, and that such a fault vertex and an isolated vertex in $G_0 \setminus F_0$, if any, may not exist simultaneously. Under these observations, we start the proof with three easy subcases. First, suppose that the fault set F_0 contains at most one vertex. Then, $G_0 \oplus G_1 \setminus F$ is matchable by a trivial p-matching composed of cross edges. Second, suppose that there exists a vertex z in $G_0 \setminus F_0$ such that $I_{G_0}(z) \subseteq F_0$. In this subcase, if we let $F'_0 = (F_0 \setminus I_{G_0}(z)) \cup \{z\}$, then $|F'_0| = |F_0| - (m-1) + 1 = m - 2$ and $G_0 \setminus F'_0$ is matchable. If $G_0 \setminus F'_0$ has a perfect matching M_0, we have a desired almost perfect matching $M_0 \cup M_1 \cup \{(z, \bar{z})\}$ for an almost perfect matching M_1 in $G_1 \setminus \{\bar{z}\}$ (remember that $|F_{01}| = 0$). Otherwise, $G_0 \setminus F'_0$ has an almost perfect matching M_0 with one unmatched vertex r, leading us to a desired perfect matching $M_0 \cup M_1 \cup \{(z, \bar{z}), (r, \bar{r})\}$, including a perfect matching M_1 in $G_1 \setminus \{\bar{z}, \bar{r}\}$. Third, assume that there exists a fault vertex v in F_0 with $I_{G_0}(v) \subseteq F_0$. Then, $G \setminus F$, which is identical to $G \setminus F'$ with $F' = F \setminus I_{G_0}(v)$, is matchable since F' is not large in size enough to preclude matchings in G.

The remaining subcase assumes that the fault set F_0 has at least two fault vertices, and that there is no vertex, whether it is in F_0 or not, all of whose incident edges belong to F_0. Again, for the simplicity of the proof, we build a fault set $F'_0 = F_0 \setminus \{v_f, w_f\}$ by removing two fault vertices v_f and w_f from F_0 as follows:

(i) If $G_0 \setminus F_0$ has two isolated vertices, then v_f and w_f are set to the two fault vertices that are respectively adjacent to them via fault-free edges in G_0, which must exist by the current subcase's assumption.

(ii) If $G_0 \setminus F_0$ has only one isolated vertex, then v_f is set to the fault vertex that is adjacent to it via a fault-free edge in G_0, and w_f is chosen arbitrarily from the remaining fault vertices in F_0.

(iii) If there is no isolated vertex in $G_0 \setminus F_0$ but an arbitrarily chosen fault vertex v_f happens to be isolated in $G_0 \setminus (F_0 \setminus v_f)$, then w_f is set to the fault vertex that is adjacent to it via a fault-free edge in G_0.

Then, the new fault set $F'_0 = F_0 \setminus \{v_f, w_f\}$ with $|F'_0| = 2m - 6$ is conditional in G_0. Now, we have two cases.

Case 3.1: $G_0 \setminus F'_0$ is matchable. If $G_0 \setminus F'_0$ has a perfect matching, $G_0 \setminus F_0$ has a perfect matching or a matching with two unmatched vertices. If $G_0 \setminus F'_0$ has an almost perfect matching, $G_0 \setminus F_0$ has an almost perfect matching or a matching with three unmatched vertices. In other words, $G_0 \setminus F_0$ has a matching M_0 with up to three unmatched vertices. So, M_0 and the corresponding free cross edge of each unmatched vertex, together with a proper p-matching in G_1, build a desired matching.

Case 3.2: $G_0 \setminus F'_0$ is not matchable. Again, we have two possibilities (i) and (ii), enumerated in the proof of ***Case 2.2***. Suppose that one of v_f and w_f in the definition of F'_0, say, v_f is black. Then, $F''_0 = (F'_0 \setminus s) \cup \{v_f\}$ is also conditional in G_0 and hence $G_0 \setminus F''_0$ is matchable, leading to ***Case 3.1***. Thus, from now on, we only consider white vertices as v_f and w_f. Furthermore, we also claim that there are two black vertices w_1 and w_2 in G_0 such that $G_1 \setminus \{\bar{w}_1, \bar{w}_2, x, y\}$ is matchable for any two vertices x and y in $G_1 \setminus \{\bar{w}_1, \bar{w}_2\}$. The claim is trivial for $m \geq 6$; when $m = 5$, it suffices to choose two black vertices w_1 and w_2 in such a way that $(\bar{w}_1, \bar{w}_2)$ is an edge of G_1, as in ***Case 2.2***, since $\{\bar{w}_1, \bar{w}_2, x, y\}$ becomes conditional in G_1 and hence $G_1 \setminus \{\bar{w}_1, \bar{w}_2, x, y\}$ is matchable. The existence of such a choice is due to the fact that $\bar{w}_1$ and $\bar{w}_2$ are selected from eight vertices in G_1, but the size of an independent set in G_1 is at most six. Thus, the claim is proved.

Now, by Lemma 5, $G_0 \setminus (F'_0 \cup \{w_1, w_2\})$ has a perfect matching M'_0 for the black vertices w_1 and w_2. If $(v_f, w_f) \in M'_0$, $G_0 \setminus (F_0 \cup \{w_1, w_2\})$ has a perfect matching M_0, and we have a desired matching $M_0 \cup M_1 \cup \{(w_1, \bar{w}_1), (w_2, \bar{w}_2)\}$ for a perfect matching M_1 in $G_1 \setminus \{\bar{w}_1, \bar{w}_2\}$. If $(v_f, w_f) \notin M'_0$, $G_0 \setminus (F_0 \cup \{w_1, w_2\})$ has a matching M_0 with two unmatched vertices r_1 and r_2, again, with free edges $(r_1, \bar{r}_1)$ and $(r_2, \bar{r}_2)$. Then, we have a desired matching $M_0 \cup M_1 \cup \{(w_1, \bar{w}_1), (w_2, \bar{w}_2), (r_1, \bar{r}_1), (r_2, \bar{r}_2)\}$ for a perfect matching M_1 in $G_1 \setminus \{\bar{w}_1, \bar{w}_2, \bar{r}_1, \bar{r}_2\}$. This completes the entire proof. $\square$

Finally, we present the statement about the CSMP number of restricted HL-graphs in dimension five or higher, which is a direct consequence of Lemma 2 and Theorem 2.

Corollary 5. *For an m-dimensional restricted HL-graph G^m, (a) $smp_1(G^m) = 2m-3$ for $m \geq 6$, and (b) $smp_1(G^5) = 6$ or 7.*

4. Concluding Remarks

In this paper, we have studied the problem of strong matching preclusion under the condition that no isolated vertex is created in a given graph as a result of vertex and/or edge faults. After briefly discussing some fundamental classes of graphs in view of the considered matching preclusion, we have rigorously investigated the CSMP number for the class of restricted HL-graphs, which include most nonbipartite hypercube-like networks found in the literature, completing the four types of matching preclusion numbers (refer to Table 1 for a summary). While the CSMP number has been revealed for all dimensional restricted HL-graphs, the minimum CSMP sets are currently known only in some low dimension. Determining every minimum CSMP set of a higher dimensional graph is left as a future research.

Acknowledgment

This research was supported by Basic Science Research Program through National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (grant no. 2011-0004419) and Industrial Strategic Technology Development Program funded by the Ministry of Knowledge Economy of Korea (grant no. 10035261-2012-01).

Appendix: Two fundamental lemmas for the proof of Lemma 5

Let C_n be a cycle graph with n vertices. Then, for any graph $G = C_n \oplus C_n$ that is bipartite, the following two lemmas hold. Implicitly, we assume that the edges of the bipartite graph G are between a set of black vertices and a set of white vertices.

Lemma 7. *Let G be a bipartite graph in the form of $C_n \oplus C_n$ with even $n \geq 4$. Then, $G \setminus F$ has a perfect matching for any fault set F consisting of two black vertices x_1, x_2 and two white vertices y_1, y_2.*

Proof. The theorem holds trivially when $n = 4$, where G is isomorphic to the 3-dimensional hypercube. So we assume that $n \geq 6$. Let $G = G_0 \oplus G_1$, where the two sides G_0 and G_1 are isomorphic to C_n. In the proof, remember that for any $u \in V(G_0)$, $\bar{u}$ is black if and only if u is white. There are four cases depending on which side contains the fault vertices.

Case 1: $x_1, x_2, y_1, y_2 \in V(G_0)$. In this case, $G_0 \setminus \{x_1, x_2\}$ consists of two odd paths P_1 and P_2, each with one more white vertex. If y_1 and y_2 belong to different paths, say, y_1 to P_1 and y_2 to P_2, it is clear that each of $P_1 \setminus y_1$ and $P_2 \setminus y_2$ has perfect matchings, and so does $G \setminus F$. If y_1 and y_2 belong to the same path, say, P_1, then $P_1 \setminus \{y_1, y_2\}$ has a unique odd path P_1' and possibly some even path(s). Let s be a black vertex in P_1' and t be a white vertex in P_2. Then, $P_1' \setminus s$ and $P_2 \setminus t$ have perfect matchings and thus $G_0 \setminus (F \cup \{s, t\})$ has a perfect matching M_0. Clearly, $G_1 \setminus \{\bar{s}, \bar{t}\}$ also has a perfect matching M_1. Hence, $G \setminus F$ has a perfect matching $M_0 \cup M_1 \cup \{(s, \bar{s}), (t, \bar{t})\}$.

Case 2: $x_1, x_2, y_1 \in V(G_0)$ and $y_2 \in V(G_1)$. $G_0 \setminus \{x_1, x_2, y_1\}$ has a unique odd path P_1 with one more white vertex since $G_0 \setminus \{x_1, y_1\}$ consists of all even path(s). For a white vertex s in P_1, $G_0 \setminus \{x_1, x_2, y_1, s\}$ and $G_1 \setminus \{y_2, \bar{s}\}$ have perfect matchings M_0 and M_1, respectively. So, $G \setminus F$ also has a perfect matching $M_0 \cup M_1 \cup \{(s, \bar{s})\}$.

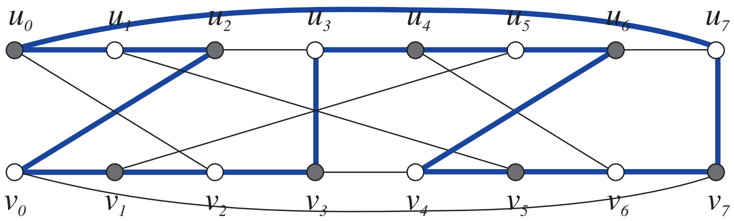
Case 3: $x_1, x_2 \in V(G_0)$ and $y_1, y_2 \in V(G_1)$. In this case, $G_0 \setminus \{x_1, x_2\}$ has two odd paths P_1 and P_2, each with one more white vertex. Also, $G_1 \setminus \{y_1, y_2\}$ has two odd paths R_1 and R_2, each with one more black vertex. We claim that there exist two white vertices s in P_1 and t in P_2 such that each of their neighbors $\bar{s}$ and $\bar{t}$ belongs to a different path, say $\bar{s}$ to R_1 and $\bar{t}$ to R_2. Then, all $P_1 \setminus s$, $P_2 \setminus t$, $R_1 \setminus \bar{s}$, and $R_2 \setminus \bar{t}$ have perfect matchings, and their union combined with $\{(s, \bar{s}), (t, \bar{t})\}$ becomes a perfect matching of $G \setminus F$.

There are two subcases. If there exist two white vertices s_1 and s_2 in the path P_1 such that each of their neighbors belong to a different path, say, $\bar{s}_1$ to R_1 and $\bar{s}_2$ to R_2, then it suffices to select an arbitrary white vertex t in P_2 and let s_2 be s if $\bar{t}$ is in R_1 or let s_1 be s if $\bar{t}$ in R_2. Otherwise, the neighbor of every white vertex in P_1 belongs to the same path, say, R_1. Since there must exist at least one white vertex in P_2, say, t, such that $\bar{t}$ is in R_2, it is enough to choose an arbitrary white vertex s in P_1 and t. Thus, the claim is proved.

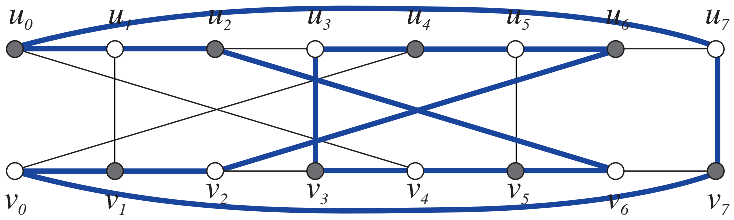
Case 4: $x_1, y_1 \in V(G_0)$ and $x_2, y_2 \in V(G_1)$. In this last case, each of $G_0 \setminus \{x_1, y_1\}$ and $G_1 \setminus \{x_2, y_2\}$ has a perfect matching, and so does $G \setminus F$. This completes the proof. □

Lemma 8. *Any bipartite graph in the form of $C_8 \oplus C_8$ has a hamiltonian cycle.*

Proof. Let $G = G_0 \oplus G_1$ be a bipartite graph, in which G_0 and G_1, isomorphic to C_8, have hamiltonian cycles $C_0 = (u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_0)$ and $C_1 = (v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_0)$, respectively. We consider two cases. If $\bar{u}_{i+1}$ is either v_{j+1} or v_{j-1} for some i such that $\bar{u}_i = v_j$ (we conveniently assume that the arithmetic on the indices is done modulo 8), it is trivial to construct a hamiltonian cycle for G. Suppose that there are no such pairs of edges. Let $\bar{u}_3 = v_3$ without loss of generality, implying that $\{\bar{u}_2, \bar{u}_4\} \cap \{v_2, v_4\} = \emptyset$, and thus $\{\bar{u}_2, \bar{u}_4\} = \{v_0, v_6\}$ due to the bipartiteness of G. If $\bar{u}_2 = v_0$ and $\bar{u}_4 = v_6$, the condition $\bar{u}_4 = v_6$ forces $\bar{u}_5 = v_1$, which, in turn, forces $\bar{u}_6 = v_4$, and so on. This sequence of conditions results in a unique graph illustrated in Figure 7(a), which has a hamiltonian cycle $(u_0, u_1, u_2, v_0, v_1, v_2, v_3, u_3, u_4, u_5, u_6, v_4, v_5, v_6, v_7, u_7, u_0)$. Similarly, the other case of $\bar{u}_2 = v_6$ and $\bar{u}_4 = v_0$ also leads to a unique graph shown in Figure 7(b), which is in fact isomorphic to that in Figure 7(a) under the mapping f with $f(u_i) = u_i$ for $0 \leq i \leq 7$, $f(v_j) = v_{6-j}$ for $0 \leq j \leq 6$, and $f(v_7) = v_7$. □



(a) When $\bar{u}_2 = v_0$ and $\bar{u}_4 = v_6$.



(b) When $\bar{u}_2 = v_6$ and $\bar{u}_4 = v_0$.

Figure 7: Two subcases in the second case of the proof of Lemma 8.

References

[1] R.E.L. Aldred, R.P. Anstee, and S.C. Locke, "Perfect matchings after vertex deletions," *Discrete Mathematics* **307**, pp. 3048–3054, Nov. 2007.

[2] P. Bonneville, E. Cheng, and J. Renzi, "Strong matching preclusion for the alternating group graphs and split-stars," *Journal of Interconnection Networks* **12(4)**, pp. 277–298, Dec. 2011.

[3] R.C. Brigham, F. Harary, E.C. Violin, and J. Yellen, "Perfect-matching preclusion," *Congressus Numerantium* **174**, pp. 185–192, 2005.

[4] E. Cheng, L. Lesniak, M.J. Lipman, and L. Lipták, "Conditional matching preclusion sets," *Information Sciences* **179(8)**, pp. 1092–1101, Mar. 2009.

[5] E. Cheng, L. Lipták, M.J. Lipman, and M. Toeniskoetter, "Conditional matching preclusion for the alternating group graphs and split-stars," *International Journal of Computer Mathematics* **88(6)**, pp. 1120–1136, 2011.

[6] E. Cheng, L. Lipták, and D. Sherman, "Matching preclusion for the (n, k)-bubble-sort graphs," *International Journal of Computer Mathematics* **87(11)**, pp. 2408–2418, 2010.

[7] E. Cheng and L. Lipták, "Matching preclusion for some interconnection networks," *Networks* **50(2)**, pp. 173–180, Sep. 2007.

[8] D.R. Guichard, "Perfect matchings in pruned grid graphs," *Discrete Mathematics* **308**, pp. 6552–6557, Dec. 2008.

[9] S.-Y. Hsieh and C.-W. Lee, "Conditional edge-fault hamiltonicity of matching composition networks," *IEEE Trans. on Parallel and Distributed Systems* **20(4)**, pp. 581–592, Apr. 2009.

[10] P.-L. Lai, J.J.M Tan, C.-H. Tsai, and L.-H. Hsu, "The diagnosability of the matching composition network under the comparison diagnosis model," *IEEE Trans. on Computers* **53(8)**, pp. 1064–1069, Aug. 2004.

[11] J.-H. Park, "Matching preclusion problem in restricted HL-graphs and recursive circulant $G(2^m, 4)$," *Journal of KIISE* **35(2)**, pp. 60–65, Feb. 2008.

[12] J.-H. Park and I. Ihm, "Strong Matching Preclusion," *Theoretical Computer Science* **412(45)**, pp. 6409–6419, Oct. 2011.

[13] J.-H. Park, H.-C. Kim, and H.-S. Lim, "Fault-hamiltonicity of hypercube-like interconnection networks," in *Proc. IEEE International Parallel and Distributed Processing Symposium IPDPS 2005*, Denver, Apr. 2005.

[14] J.-H. Park, H.-C. Kim, and H.-S. Lim, "Many-to-many disjoint path covers in the presence of faulty elements," *IEEE Trans. on Computers* **58(4)**, pp. 528–540, Apr. 2009.

[15] J.-H. Park and S.H. Son, "Conditional matching preclusion for hypercube-like interconnection networks," *Theoretical Computer Science* **410(27-29)**, pp. 2632–2640, June 2009.

[16] A.S. Vaidya, P.S.N. Rao, S.R. Shankar, "A class of hypercube-like networks," in *Proc. of the 5th IEEE Symposium on Parallel and Distributed Processing SPDP 1993*, pp. 800–803, Dec. 1993.

[17] S. Wang, R. Wang, S. Lin, and J. Li, "Matching preclusion for k-ary n-cubes," *Discrete Applied Mathematics* **158(18)**, pp. 2066–2070, Nov. 2010.