Low Degree Approximation of Surfaces for Revolved Objects*

Chandrajit L. Bajaj
Department of Computer Sciences
Purdue University
West Lafayette, Indiana 47907
e-mail : bajaj@cs.purdue.edu

Insung Ihm
Department of Computer Science & Engineering
Sogang University
Seoul, Korea
e-mail : ihm@ccs.sogang.ac.kr

Abstract

We present a method for generating low degree C^k-continuous piecewise algebraic surfaces for revolved objects. The approximating pieces are implicitly defined algebraic surfaces whose profile curves can be obtained algebraically or parametrically from digitized points. We show that degree d surface patches can be used for approximations with inter-patch C^k continuity as high as $k = \lfloor \frac{(d+2)^2-12}{8} \rfloor$ for even d, and $k = \lfloor \frac{(d+1)(d+3)-12}{8} \rfloor$ for odd d. As an example, we construct C^1 cubic surfaces and C^2 quartic surfaces for revolved objects from digitized profile curves.

Keywords : Algebraic surface, approximation, C^k continuity, curves, digitized data, polynomial, revolution

1 Introduction

Algebraic curves and surfaces can be represented in an implicit form, and sometimes also in a parametric form. The implicit form of a real algebraic surface in $\mathbb{R}^3$ is

$$f(x, y, z) = 0 \tag{1}$$

where f is a polynomial with coefficients in $\mathbb{R}$. The parametric form, when it exists, for a real algebraic surface in $\mathbb{R}^3$ is

$$\begin{aligned} x &= \frac{f_1(s,t)}{f_4(s,t)} \\ y &= \frac{f_2(s,t)}{f_4(s,t)} \\ z &= \frac{f_3(s,t)}{f_4(s,t)} \end{aligned} \tag{2}$$

where the f_i are again polynomials with coefficients in $\mathbb{R}$. The *algebraic degree* of an algebraic curve or surface (in implicit or parametric form) is the $maximum$ degree of any defining polynomial. The *geometric degree* of an algebraic plane curve or surface (in implicit or parametric form) is the $maximum$ possible number of intersections with any line. The intersections are counted with respect to a plane for algebraic space curves [2].

This paper presents two main ideas to be used in fitting low degree, piecewise algebraic surfaces (in the implicit or parametric form) to data sampled from arbitrary boundary surfaces of solids of revolution. One is the use of degree restricted bases for the piecewise approximation of the generating curve of revolution surfaces to yield approximating surfaces of the same algebraic degree as the degree of the piecewise curves. The other new idea arises in the development and use of C^k implicit algebraic splines for degree restricted interpolation and approximation of generating curves. While traditional fitting schemes are predominantly based on piecewise parametric representations[5, 6], we show here that implicit representations are also quite appropriate and in fact better equipped for restrictions on the bases and the degrees of the involved polynomials.

From Bezout's theorem[11], we realize that the intersection of two implicit surfaces of algebraic degree d can be a curve of geometric degree $O(d^2)$. Furthermore the same theorem implies that the intersection of two parametric surfaces of algebraic degree d can be a curve of degree $O(d^4)$. Hence, while the potential singularities of the space curve defined by the intersection of two implicit surfaces defined by polynomials of degree d can be as many as $O(d^4)$, the potential singulari-

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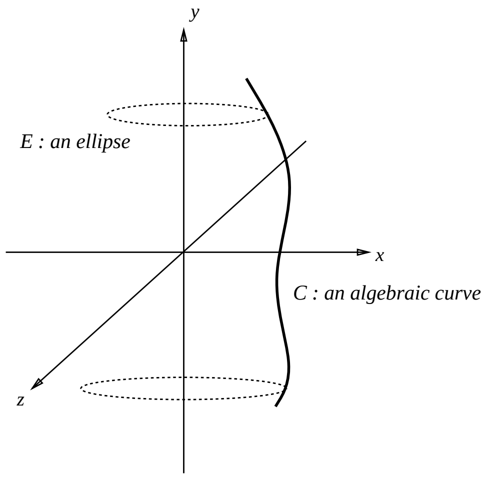


Figure 1: Revolution of an Algebraic Curve along an Ellipse

ties of the space curve defined by the intersection of two parametric surfaces defined by polynomials of degree d can be as many as $O(d^8)$[2]. Hence keeping the degree of fitting surfaces as low as possible benefits both the efficiency and the robustness of post processing for modeling and display[1].

The rest of this paper is as follows. Section 2 characterizes the appropriate degree restricted bases for implicit and parametric algebraic curves which would yield revolution surfaces of the same algebraic degree as the degree of the curves. Section 3 characterizes C^k continuous piecewise surfaces of revolution and their construction from sampled data points. Section 4 describes the development and details for constructing cubic implicit algebraic C^1 and C^2 splines for approximating generating curves of surfaces of revolution.

2 Surfaces of Revolution

2.1 Algebraic Surfaces of Revolution

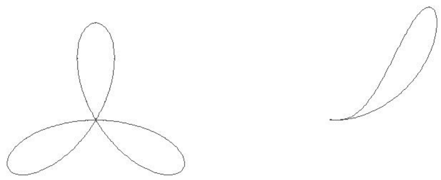
Consider an algebraic surface which is obtained by revolving an algebraic curve $f(x, y) = 0$ (on the xy plane) around the y axis. (See Figure 1.) Rather than restricting ourselves to a circular rotation, we consider a more general elliptic revolution where the rotation path is described by an ellipse $E : x^2 + \frac{z^2}{\alpha^2} = \{r(y)\}^2$ with $\alpha > 0$. Here, $r(y)$ is the x coordinate of the point (x, y) on the curve $C : f(x, y) = 0$.

Now, the surface that results from revolving C along E is specified as "$x^2 + \frac{z^2}{\alpha^2} = \{r(y)\}^2$ *subject to* $f(r(y), y) = 0$." The equation $F(x, y, z) = 0$ of the surface S, hence, becomes $F(x, y, z) = f(\sqrt{x^2 + \frac{z^2}{\alpha^2}}, y) = 0$ where $F(x, y, z)$ is not necessarily *algebraic* due to introduction of the square root. By allowing only even-powered x's (x^0, x^2, x^4, $\cdots$) in $f(x, y)$, we can force $F(x, y, z)$ to be algebraic. Geometrically, this restriction, imposed on the revolved curve, that maintains *algebraicity*, means that the curve $f(x, y) = 0$ is symmetric to the y axis.

For quadric curves $f(x, y) = 0$, x^2 is the only possible factor of terms in f. Hence, f includes a 4-dimensional vector space V_f^2 of polynomials over real numbers that is spanned by the basis $\{x^2, y^2, y, 1\}$. In case of cubic curves $f(x, y) = 0$, the vector space V_f^3 is spanned by the basis $\{x^2y, x^2, y^3, y^2, y, 1\}$ with dimension 6. Quartic curves $f(x, y) = 0$ can be chosen from a more abundant vector space V_f^4 of dimension 9, generated by the basis $\{x^4, x^2y^2, x^2y, x^2, y^4, y^3, y^2, y, 1\}$. The bases of vector spaces V_f^d for higher degree curves are formulated in the same fashion.

Each algebraic curve of degree d in V_f^d, revolved around an ellipse, results in an algebraic surface of the same degree. Then we naturally come to the following question : "Is a surface, generated by revolving around an ellipse an algebraic curve that is not in V_f^d, algebraic at all?" In fact, the surface *is* algebraic, though the surface's degree gets doubled. This doubling of the degree arises from the single squaring required to remove the square root from odd-powered x factors. For example, consider a circle $f(x, y) = (x - 5)^2 + (y - 5)^2 - 1 = x^2 - 10x + y^2 - 10y + 49 = 0$ of radius 1, centered at $(5, 5)$. This conic curve is not in V_f^2 because of the term $10x$. However, by moving $10x$ to the right hand side, and then squaring both sides, we can obtain a quartic curve in V_f^4 which generates a torus (of degree 4) by rotation. Intuitively, the squaring operation has an effect of putting another circle of the same shape to the other side of the y axis in order to artificially make the curve symmetric to the y axis. Any algebraic curve of degree d which is not in V_f^d can be made to be in V_f^{2d} by moving all terms with odd-powered x factors to one side, and squaring both sides.

REMARK 2.1. *Let* $C : f(x, y) = 0$ *be an algebraic curve of degree* d*, and* $E : x^2 + \frac{z^2}{\alpha^2} = \{r(y)\}^2$ *be an ellipse of a rotation path. Then, the algebraic surface* $S : F(x, y, z) = 0$*, resulting from revolving* C *around* E*, has degree* d *if* C *is symmetric around the* y *axis, or* $2d$ *otherwise.*



(a) (b)

Figure 2: Two Quartic Algebraic Curves

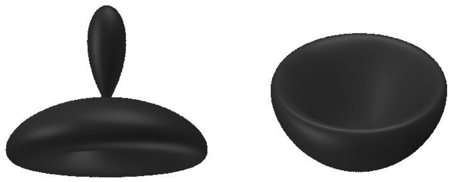
A geometric interpretation to Remark 2.1 is as follows : Consider a line on the xy plane parallel to the x axis. This line intersects with C at most d times. Now, imagine the intersection between the line and S. When C is symmetric, the number of intersection remains the same. However, if C is not symmetric, the number of intersection is doubled up because C, rotated by 180 degrees, creates the same number of line-curve intersections.

It is important to understand that, the degrees of freedom, in choosing a curve $f(x, y) = 0$ of degree d from V_f^d, is $\mathsf{dim}(V_f^d) - 1$ where $\mathsf{dim}(*)$ is the dimension of a vector space. Since all the polynomials on a line in V_f^d that passes through f and 0 describe the same curve, we have one less than $\mathsf{dim}(V_f^d)$ degrees of freedom. It is not hard to come up with the expression for $\mathsf{dim}(V_f^d)$:

$$\mathsf{dim}(V_f^d) = \begin{cases} \frac{(d+2)^2}{4} & \text{if } d \text{ is even} \\ \frac{(d+1)(d+3)}{4} & \text{if } d \text{ is odd} \end{cases}$$

In many situations as will be shown later, the curve $f(x, y) = 0$ is to be designed such that it satisfies given geometric requirements. We are interested in designing piecewise curves from given digitized data, and revolving them in a complicated manner to model some class of objects with low degree algebraic surfaces. It will be explained below how the degrees of freedom in piecewise algebraic curves of a given degree limit the geometric continuity between them.

EXAMPLE 2.1. Figure 2 (a) and (b) displays two quartic algebraic curves $(x^2 + y^2)^2 + 3x^2y - y^3 = 0$ and $x^4 + x^2y^2 - 2x^2y - xy^2 + y^2 = 0$, respectively [12]. The curves, after rotation, result in algebraic surfaces of degree 4 and 8, respectively, and shown in Figure 3 (a) and (b).



(a) (b)

Figure 3: Degree 4 and 8 Algebraic Surfaces of Revolution

2.2 Parametric Surfaces of Revolution

Now, we get to a question : "Is it also possible to find a restricted bases of *rational parametric* curves that result in *rational parametric* surfaces of the *same geometric* degree after revolution around an axis?" Consider a rational parametric curve of degree d

$$C(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} \frac{x(t)}{w(t)} \\ \frac{y(t)}{w(t)} \end{pmatrix}$$

where the degrees of the polynomials $x(t)$, $y(t)$, and $w(t)$ are at most d. The surface obtained by revolving $C(t)$ around y-axis along an ellipse $E : x^2 + \frac{z^2}{\alpha^2} = \{r(y)\}^2$ with $\alpha > 0$ can be represented as $F(s, t) = (X(s, t), Y(s, t), Z(s, t))$, where

$$\begin{aligned} X(s,t) &= \frac{2s}{1+s^2}\frac{x(t)}{w(t)} \\ Y(s,t) &= \frac{y(t)}{w(t)} \\ Z(s,t) &= \frac{\alpha(1-s^2)}{1+s^2}\frac{x(t)}{w(t)}. \end{aligned}$$

First, this representation answers that the revolved surface is always rational parametric. Then, the second question on the degree of $F(s, t)$ must be answered. We are interested in lowering both the *algebraic* degree in the polynomials in $F(s, t)$ and the *geometric* degree of $F(s, t)$ (the maximum possible intersection of $F(s, t)$ with a line). In construction of rational parametric revolved surfaces, we follow the same path we did in the previous subsection. From Remark 2.1, we know that an algebraic curve of degree d generates an algebraic surface of the same degree only when it is symmetric around an axis. Since every rational parametric curve of degree d is an algebraic curve of degree d, we are led to the fact that $F(s, t)$ is of degree d if $C(t)$ is symmetric around the y-axis.

A rational parametric curve is symmetric if there is a parametrization $C(t) = (X(t), Y(t)) = (\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)})$ such that $X(t) = -X(-t)$ and $Y(t) = Y(-t)$. That is,

$$\frac{x(t)}{w(t)} = -\frac{x(-t)}{w(-t)} \quad (3)$$

$$\frac{y(t)}{w(t)} = \frac{y(-t)}{w(-t)} \quad (4)$$

The above conditions are met if *either* $x(t)$ is an odd function (all the terms with nonzero coefficients are odd-powered), and $y(t)$, $w(t)$ are even functions (all the terms with nonzero coefficients are even-powered), *or* $x(t)$ is an even function, and $y(t)$, $w(t)$ are odd functions. It is not difficult to see that the polynomials in the second case can be converted into the first case polynomials by multiplying t to both numerator and denominator, and vice versa. In fact, any polynomials that satisfies the conditions (3) and (4) fall in the above two categories.

LEMMA 2.1. *Let $x(t)$, $y(t)$, and $w(t)$ be polynomials in t such that $x(t)$ and $w(t)$ are relatively prime, and $y(t)$ and $w(t)$ are relatively prime. Then, $x(t)$ is an odd function, and $y(t)$, $w(t)$ are even functions if and only if $\frac{x(t)}{w(t)} = -\frac{x(-t)}{w(-t)}$ and $\frac{y(t)}{w(t)} = \frac{y(-t)}{w(-t)}$.*

PROOF : See [3]. □

From now on, we assume that $x(t)$ is an odd function, and $y(t)$ and $w(t)$ are even functions without loss of generality. Since a degree d curve $C(t) = (X(t), Y(t)) = (\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)})$ is symmetric around y-axis, the surface made by revolving it around y-axis is a surface of geometric degree d. The surface equation $F(s,t)$ given above is represented by degree $d+2$ polynomials. In [3] we show it is possible to reduce the algebraic degree of the parametric surface equations to d by applying a transformation to $F(s,t)$.

REMARK 2.2. *Let $C : C(t) = (\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)})$ be a rational parametric curve of degree d where $x(t)$ is an odd function, and $y(t)$, $w(t)$ are even functions, and $E : x^2 + \frac{z^2}{\alpha^2} = \{r(y)\}^2$ be an ellipse of a rotation path. Then, the algebraic surface $S : F(s,t) = (X(s,t), Y(s,t), Z(s,t))$ in the rational parametric form, resulting from revolving C around E, has geometric degree d, and can be parameterized in the way that $X(s,t)$, $Y(s,t)$, and $Z(s,t)$ are degree d rational polynomials.*

The class of the above rational parametric curves contains symmetric parametric curves that intersect with y-axis. The set of all such curves is only a proper subset of all symmetric parametric curves. Another interesting class of symmetric rational parametric curves is defined as $C(t) = (X(t), Y(t)) = (\frac{x(t)}{w(t)}, \frac{y(t)}{w(t)})$ such that $X(t) = -X(-\frac{1}{t})$ and $Y(t) = Y(-\frac{1}{t})$[1]. It still remains open how to specify all the bases of symmetric rational parametric curves of a given degree.

EXAMPLE 2.2. Recall the "three-leaf clover" in Example 2.1. Its parametric form is $C(t) = (\frac{t^3-3t}{t^4+2t^2+1}, \frac{t^4-3t^2}{t^4+2t^2+1})$. After circular revolution and the above mentioned reparametrization, the quartic surface is $F(u,v) = (\frac{u(u^2+v^2-3)}{(u^2+v^2)^2+2(u^2+v^2)+1}, \frac{(u^2+v^2)^2-3(u^2+v^2)}{(u^2+v^2)^2+2(u^2+v^2)+1}, \frac{v(u^2+v^2-3)}{(u^2+v^2)^2+2(u^2+v^2)+1})$ and shown in Figure 3 (a).

3 Construction of Piecewise C^k Continuous Revolved Objects

So far we have discussed about revolution of a single algebraic curve, represented in either the implicit or the parametric form. A revolved object with a complicated shape, however, cannot be modeled by rotating only one low degree curve. Instead, it is more appropriate to approximate a revolved object using surface patches meeting together with some order of geometric continuity. Hence, the revolved object design problem leads to the following basic problem: design piecewise C^k continuous algebraic curve segments, with restricted bases.

In this paper we focus on the design of piecewise C^k continuous *implicitly represented algebraic* curve segments.[2] Designing with parametric splines is explained in [5] in detail. Also, we shall exhibit that designing with symmetric (restricted bases) implicit algebraic curves is no more difficult than with the complete basis. The corresponding case of designing with symmetric parametric curves does not directly follow from the general parametric case and is a an open problem for further research.

3.1 Algebraic Curves and Geometric Continuity

In this subsection, we describe how to compute two algebraic curves that meet with C^k continuity at a point.

[1] For example, a hyperbola is in this class.

[2] From now on, by "algebraic", we mean "implicit algebraic".

First of all, we assume the geometric information about a point p is expressed in terms of a (truncated) power series $C(t)$ of degree k, where $C(t) = (x(t), y(t)) = p + c_1 t + c_2 t^2 + \cdots + c_k t^k$, and $C(0) = p$. This truncated power series approximates the local geometric property (up to order k) of a curve about the point within a radius of convergence. (We will discuss later how this power series is computed.)

Now, given a (truncated) formal power series $C(t)$ about a point p, we find an algebraic curve $f(x, y) = 0$ whose power series expansion at p is the same as $C(t)$ at p. If all terms upto degree k agree for $f(x, y) = 0$ and $C(t)$ at p then $f(x, y) = 0$ is considered to meet $C(t)$ with C^k continuity at p. Let $f(x, y) = \sum_{i+j \le d} a_{ij} x^i y^j = 0$ be an algebraic curve of degree d, and

$$
\begin{aligned}
C(t) &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\
&= \begin{pmatrix} p_x + c_{1x} t + c_{2x} t^2 + \cdots + c_{kx} t^k \\ p_y + c_{1y} t + c_{2y} t^2 + \cdots + c_{ky} t^k \end{pmatrix}
\end{aligned}
$$

be a given parametric polynomial such that $C(0) = (p_x, p_y) \equiv p$. The relations on the coefficients of $f(x, y)$ can be extracted by repeatedly differentiating $f(C(t))$ up to order k, making all the derivatives vanish at $t = 0$ [7]. The first few partial derivatives are :

$$
\begin{aligned}
f(C(t))\,|_{t=0} &= f(p) = 0 \\
\frac{df(C(t))}{dt}\,|_{t=0} &= f_x(p)x'(0) + f_y(p)y'(0) \\
&= c_{1x} f_x(p) + c_{1y} f_y(p) = 0 \\
\frac{d^2 f(C(t))}{dt^2}\,|_{t=0} &= f_{xx}(p)x'(0)^2 \\
&\quad + 2 f_{xy} x'(0) y'(0) \\
&\quad + f_{yy}(p) y'(0)^2 \\
&\quad + f_x(p) x''(0) + f_y(p) y''(0) \\
&= c_{1x}^2 f_{xx}(p) + 2 c_{1x} c_{1y} f_{xy}(p) \\
&\quad + c_{1y}^2 f_{yy}(p) + c_{2x} f_x(p) \\
&\quad + c_{2y} f_y(p) = 0 \\
&\cdots
\end{aligned}
$$

For each derivative of $f(C(t))$, a linear equation in terms of the unknown coefficients a_{ij} of f is generated, hence, any solution of the homogeneous linear system of $k + 1$ equations becomes coefficients of algebraic curves of degree d meeting $C(t)$ with C^k continuity. Since an algebraic curve segment needs to satisfy the C^k conditions at both end points, $2k + 2$ linear constraints must be satisfied. Hence, in order for an algebraic curve of degree d to exist, d must be chosen such that $\binom{d+2}{2} - 1 \ge 2k + 2$, that is, the number of the degrees of freedom in coefficients of the curve is greater than or equal to the constraints for C^k continuity. Exactly the same process is applied for symmetric implicit algebraic curves of degree d with restricted bases, with the difference being that the number of degrees of freedom is given by $\mathsf{dim}(V_f^d) - 1$ as shown in section 2.1.

3.2 Computation of a Truncated Power Series

There are various forms of divided-difference methods that extract geometric information around a point, from a given list of points [5]. In our case, we choose a parabola to locally approximate the points about a junction point, and take out tangential information from the parabola. The junction points themselves are for now, computed using the dynamic programming scheme in [8] which minimizes the error for a piecewise linear approximation (with fixed number of segments) to a set of digitized points. Consider a sequence of points $\cdots, p_{i-2}, p_{i-1}, p_i, p_{i+1}, p_{i+2}, \cdots$ around the junction point p_i and an imaginary power series $C(t)$ from which, we assume, the digitized points near p_i arise, and whose parameter value is $t = 0$ for p_i. Then, the tangent vector of $C(t)$ at $t = 0$ can be approximated by the approximation :

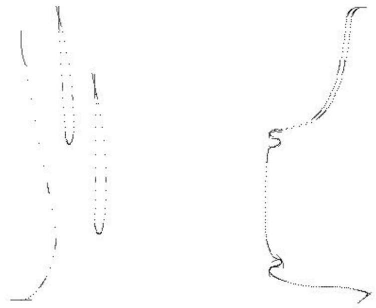
$$
\begin{aligned}
C'(0) &\approx \frac{\sigma_i}{\mathsf{dist}(p_i, p_{i+1})}(p_{i+1} - p_i) \\
&+ \frac{1 - \sigma_i}{\mathsf{dist}(p_{i-1}, p_i)}(p_i - p_{i-1})
\end{aligned}
$$

where $\sigma_i = \frac{\mathsf{dist}(p_{i-1}, p_i)}{\mathsf{dist}(p_i, p_{i+1}) + \mathsf{dist}(p_{i-1}, p_i)}$ and $\mathsf{dist}(*, *)$ is the distance between two points.

Repeatedly applying this approximation formula, we introduce a divided-difference :

$$
\Delta^j p_l = \begin{cases} p_l & \text{if } j = 0 \\ \frac{1}{j}\left(\frac{\sigma_l}{\mathsf{dist}(p_l, p_{l+1})}(p_{l+1} - p_l) \right. & \\ \left. + \frac{1 - \sigma_l}{\mathsf{dist}(p_{l-1}, p_l)}(p_l - p_{l-1})\right) & \text{if } j > 0 \end{cases}
$$

Using this divide-difference operator, a truncated power series is represented as $C_i(t) = \Delta^0 p_i + \Delta^1 p_i t + \Delta^2 p_i t^2 + \cdots + \Delta^k p_i t^k$. Note that the geometric information, stored in the coefficients of the power series is extracted from a sequence of $2k+1$ neighboring points, centered at the junction point. This locality in the construction of a power series enables an interactive local modeling operation.



(a) (b)

Figure 4: Digitized Engine and Goblet with Truncated Power Series

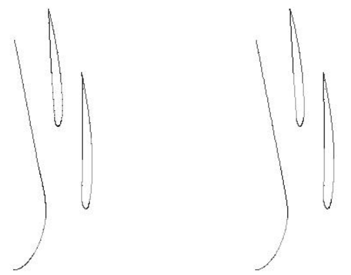
EXAMPLE 3.1. In Figure 4, two sets of digitized points are illustrated. (a) shows three lists of points that model engine parts[3], and (b) is a sequence of points that models a goblet. Each point sequence is displayed with truncated power series of order two at junction points.

3.3 Families of Algebraic Curves $f(x, y)$

In order to compute each curve segment $f_i(x, y) = 0$ that interpolates two truncated power series $C_i(t)$ and $C_{i+1}(t)$ at two end points p_i and p_{i+1}, respectively, we construct a linear system $\mathbf{M_I x} = \mathbf{0}$ where the unknowns are coefficients of $f_i(x, y) = 0$. The linear system is made of $2(k + 1)$ equations that are generated for both truncated power series. Note that the rank of $\mathbf{M_I}$ must be less than the number of unknowns for a nontrivial solution to exist. Any nontrivial solution represents an algebraic curve that meets $C_i(t)$ and $C_{i+1}(t)$ at p_i and p_{i+1}, respectively, with C^k continuity. One heuristic that we have often used is to select a nice curve segment is to generate a sequence of additional points between the end points that approximate a curve segment, and then, apply least-squares approximation to these additional points. In the case of cubic algebraic curves, in Section 4 we derive a condition on the Bernstein-Bezier coefficients of cubic curves, in either the general or the restricted basis, that guarantees a smooth single curve segment inside a given control triangle.

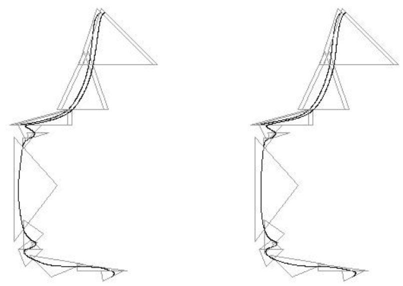
In case all possible terms of degree d are used as a basis of $f_i(x, y) = 0$, then there are $\binom{d+2}{2}$ unknowns, and hence $\binom{d+2}{2} - 1$ degrees of freedom. However, if we choose a curve from V_f^d, we have fewer degrees of freedom due to restriction in the basis. There are only $\mathsf{dim}(V_f^d) - 1$ degrees of freedom for degree d, and this number must not be less than $2(k + 2)$, the maximum possible rank for a homogeneous linear system that needs to be satisfied for order k continuity. For instance, for C^1 continuity, symmetric cubic curves are necessary, while order 2 continuity requires symmetric quartic curves.

[3]This data originated from 3D scanned engine data from NASA.



(a) (b)

Figure 5: Symmetric C^1 Cubic and C^2 Quartic Algebraic Splines

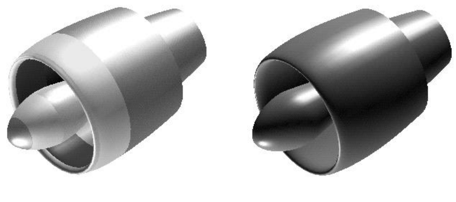


(a) (b)

Figure 6: Symmetric C^1 and Arbitrary C^2 Cubic Algebraic Splines

3.4 Piecewise C^k Continuous Revolved Objects

Figure 5 (a) displays piecewise C^1 approximation with cubic algebraic curves in the restricted basis V_f^3. Note that a symmetric cubic curve in V_f^3 can have a tangent line parallel to x-axis only at points on the y-axis. Hence, the order of geometric continuity is only 0 at the extreme junction points on the cowls around which the curve segments make vertical turnabouts. With symmetric quartic algebraic curves in V_f^4, it is possible to approximate the point data with C^2 continuity everywhere. (See Figure 5 (b).) For the goblet data, cubic curves in V_f^3, again, successfully model the data with



(a) (b)

Figure 7: C^1 Cubic and C^2 Quartic Revolved Surface Models

(a) (b)

Figure 8: C^1 Cubic and C^2 Sextic Revolved Surface Models

C^1 continuity in Figure 6 (a). Figure 6 (b) shows a C^2 approximation of the same data with cubic curves in the general basis, which, hence, may not be symmetric about the y-axis.

Once algebraic splines are constructed to fit the digitized data, their revolution surface models are easily obtained, with the appropriate surface degree bounds. C^1 approximation with cubic algebraic surfaces is shown in Figure 7 (a) and are a revolution of the cubic splines in Figure 5 (a). Quartic algebraic surfaces approximate the same object well with C^2 continuity in Figure 7 (b) and are a revolution of the quartic splines in Figure 5 (b). A C^1 cubic algebraic surface goblet is illustrated in Figure 8(a) and is obtained by revolving the symmetric cubic spline in Figure 6 (a). The C^2 goblet in Figure 8(b) is obtained by revolving the arbitrary cubic splines in Figure 6 (b), and is made of degree 6 algebraic surfaces.

4 Cubic Algebraic Splines

In this section, we focus on implicitly defined cubic algebraic curves, and give conditions on the coefficients of cubic algebraic curves that guarantee nice properties inside regions bounded by triangles. These conditions can be equally applied to cubic curves in the restricted or the general basis.

Paluszny and Patterson [9] considered a special family of implicit cubic curves which yields tangent continuous cubic splines. Our method here differs in that both tangents and curvatures are specified and the splines are not limited to be convex inside the bounding triangles. Bajaj and Xu [4] show how to construct C^3 continuous cubic algebriac splines, however their method is not directly applicable for symmetric restricted bases.

4.1 Interpolation with Cubic Algebraic Curves

A *general* [4] cubic algebraic curve in the Bernstein basis is defined as $B^3(u,v) = \sum_{i+j\leq 3} w_{ij} B^3_{ij}(u,v) = 0$. (For introduction to barycentric coordinates, see [6].) Sederberg [10] proposed to view an algebraic curve as the intersection of the explicit surface $w = B^d(u,v)$ with the plane $w = 0$, hoping to associate geometric meanings to the coefficients of the polynomial. Especially, the coefficients in the polynomial are considered as the w coordinates of the control net of a triangular Bernstein-Bézier surface patch, where the coefficient w_{ij} corresponds to the control point $b_{ij} = (\frac{i}{3}, \frac{j}{3})$ in the Bernstein basis. The coefficients w_{ij} is relative to selection of a control triangle $\mathcal{T} = (P_{00}, P_{30}, P_{03})$ in the power basis. There are ten coefficients, and since dividing the equation out by a nonzero number would not change the algebraic curve, we see that there are nine degrees of freedom. For symmetric *restricted* cubic algebraic curves in the Bernstein basis there are only five degrees of freedom. Hence, three degrees of freedom are left after C^2 interpolation with general cubic algebraic curves, and one for C^1 interpolation with restricted cubics.

4.2 Computation of Effective Cubic Algebraic Spline Curves

We describe in some detail the case of C^2 continuous general algebraic cubic splines. Computation of C^1 continuous restricted algebraic cubic splines can be achieved along similar lines. Let $C_{B_0}(t)$ and $C_{B_1}(t)$ be two truncated power series of degree two that describe geometric properties at two points π_0 and π_1, respectively. One of goals we try to accomplish is to find a triangle within which a single connected smooth segment of a cubic algebraic curve is confined such that

[4] We use the adjectives *general* and *restricted* to distinguish cubic algebraic curves in the general and the restricted bases, respectively.

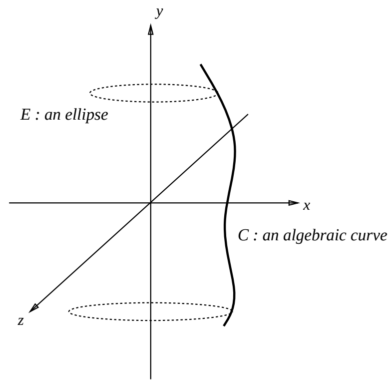


Figure 9: An Effective Spline Curve

the curve segment achieves C^2 continuity at π_0 and π_1 and subdivides the triangle into a positive and a negative space. (See Figure 4.2.)

DEFINITION 4.1. Let $\mathcal{T}$ be a triangle made of three vertices P_{00}, P_{d0}, P_{0d}. Consider a smooth curve segment of degree n on $B^d(u,v) = 0$ whose two end points are on the two sides $\overline{P_{00}P_{d0}}$ and $\overline{P_{00}P_{0d}}$. The curve segment is called *an effective algebraic spline associated with the bounding triangle* $\mathcal{T}$ if the curve segment intersects exactly once a line segment connecting P_{00} and any point on the side $\overline{P_{d0}P_{0d}}$.

The restriction imposed in the definition of an effective spline reomves disconnected curve segments, loops, unwanted extra pieces and singularities from within the bounded triangle. It also forces the spline curve segment to subdivide a bounding triangle into a positive and a negative space. The ability of finding an effective spline with a proper bounding triangle is essential in that it allows easy implementations of many geometric modeling operations [1]. A point can be easily classified as in, out, or on the boundary of an object that is made of several algebraic splines. This point-classification operation is a primitive operation to high level geometric modeling operations.

For a spline curve segment that is C^2 continuous at the end points π_0 and π_1 within the triangle $\mathcal{T}$, interpolation of the respective truncated power series at these points with a cubic polynomial generates six constraints, leaving three degrees of freedom. After solving the homogeneous linear system with ten unknowns, and six linearly independent constraints, the ten coefficients can be expressed in terms of linear functions in four free parameters λ_0, λ_1, λ_2, and λ_3. We next set up constraints on these free parameters such that for feasible values of λ_i, $i = 0, 1, 2, 3$, the curve segment is a single piece within $\mathcal{T}$. Note that the feasible values of λ_i, $i = 0, 1, 2, 3$, are those for which the triangular Bernstein-Bezier surface patch corresponding to $\mathcal{T}$ intersects the plane $w = 0$ within $\mathcal{T}$ *exactly once* and as shown in Figure 10.

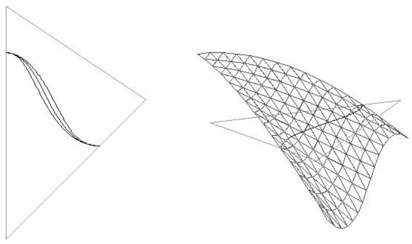
LEMMA 4.1. *Let ten coefficients w_{ij} of $B^3(u,v)$ be expressed linearly in terms of λ_j, $j = 0, 1, 2, 3$ after C^2 interpolation of $C_{B_0}(t)$ and $C_{B_1}(t)$ at π_0 and π_1, respectively, with respect to a control triangle $\mathcal{T}$. Then, there exists an effective cubic algebraic spline associated with $\mathcal{T}$ if and only if there exists some λ_j, $j = 0, 1, 2, 3$ such that the univariate cubic polynomial $G(x) \stackrel{\text{def}}{=} B^3((1-\alpha)x, \alpha x) = g_3(\alpha)x^3 + g_2(\alpha)x^2 + g_1(\alpha)x + g_0(\alpha)$ has one and only one root in $0 \le x \le 1$ for all $\alpha \in [0,1]$. The $g_i(\alpha)$, ($i = 0, 1, 2, 3$) are polynomials of degree i in α with coefficients which are linear relations on w_{ij} and hence of the free parameters λ_j, ($j = 0, 1, 2, 3$).*

PROOF : See [3]. □

Due to the limited space, we now present only the final results Details can be found in [3]. Consider the three cases where $h_i(\alpha)$, $(i = 0, 1, 2, 3)$ is a degree $3-i$ polynomial in α and a linear combination of the above $g_i(\alpha)$ polynomials. The coefficients of $h_i(\alpha)$ are linear combinations of the free parameters λ_j, $(j = 1, 2, 3)$:

- `[CASE 1]` $h_3(\alpha) = 1 > 0$, $h_2(\alpha)^2 - 3h_3(\alpha)h_1(\alpha) \le 0$, $h_0(\alpha) < 0$
- `[CASE 2]` $h_3(\alpha) = 1 > 0$, (either $h_2(\alpha) \ge 0$ or $h_1(\alpha) \le 0$), $h_0(\alpha) < 0$
- `[CASE 3]` $h_3(\alpha) = 1 > 0$, $h_2(\alpha) < 0$, $h_1(\alpha) > 0$, $h_0(\alpha) < 0$, $h_2(\alpha)^2 - 3h_3(\alpha)h_1(\alpha) > 0$, $(-27h_0(\alpha)h_3(\alpha)^2 + 9h_1(\alpha)h_2(\alpha)h_3(\alpha) - 2h_2(\alpha)^3) > 0$, $(27h_0(\alpha)^2h_3(\alpha)^2 - 18h_0(\alpha)h_1(\alpha)h_2(\alpha)h_3(\alpha) + 4h_1(\alpha)^3h_3(\alpha) + 4h_0(\alpha)h_2(\alpha)^3 - h_1(\alpha)^2h_2(\alpha)^2) > 0$

THEOREM 4.1. *Let ten coefficients w_{ij} of $B^3(u,v)$ be expressed linearly in terms of λ_j, $j = 1, 2, 3$ with $w_{00} = 1$ after C^2 interpolation of $C_{B_0}(t)$ and $C_{B_1}(t)$ at π_0 and π_1, respectively, with respect to a control triangle $\mathcal{T}$. Then, there exists an effective cubic algebraic spline associated with $\mathcal{T}$ if and only if there exists some λ_j, $j = 1, 2, 3$ such that, for all $\alpha \in [0,1]$, either* `[CASE 1]`, `[CASE 2]`, *or* `[CASE 3]` *is satisfied.*



(a) (b)

Figure 10: C^2 Continuous Cubic Algebraic Spline Curves

Theorem 4.1 generates inequality constraints whose expressions are linear, quadratic, cubic, and quartic in λ_1, λ_2, λ_3. Hence, all the feasible solutions (λ_1, λ_2, λ_3) of those constraints comprise a union of subspaces in the three dimensional $\lambda_1\lambda_2\lambda_3$ solution space bounded by linear, quadratic, cubic, or quartic algebraic surfaces. Choosing an effective cubic algebraic spline associated with a bounding triangle becomes equivalent to finding feasible points in these subspaces. In our implementation we currently use standard nonlinear numerical optimization techniques to compute feasible solutions. Given the low dimensionality of the solution space and the bounded degree of the constraints, we are currently experimenting with symbolic methods which yield a cell decomposition of the feasible region for easy solution point generation and navigation.

EXAMPLE 4.1. Figure 10(a), shows three instances of cubic algebraic curves that C^2 interpolate the two endpoint truncated power series $C_0(t) = (1 + t, t^2)$ and $C_1(t) = (t, 1 - 2t^2)$ with respect to $\mathcal{T} = ((0.0, -1.0), (1.5, 0.5), (0.0, 1.5))$. The three curves chosen from the four dimensional space are $f_0(x, y) = 0.757333x^3 - 1.19933x^2y - 0.768667x^2 + 0.534667xy^2 + 0.2xy - 0.734667x + 0.004y^3 - 0.246y^2 - 0.504y + 0.746$, $f_1(x, y) = 4.08x^3 - 7.37x^2y - 5.99x^2 + 0.06xy^2 + 0.2xy - 0.26x - 1.42y^3 - 1.67y^2 + 0.92y + 2.17$, and $f_2(x, y) = 0.421333x^3 - 0.575333x^2y - 0.240667x^2 + 0.582667xy^2 + 0.2xy - 0.782667x + 0.148y^3 - 0.102y^2 - 0.648y + 0.602$. As C^2 continuity implies, $f_i(C_j(t)) = O(t^3)$, $i = 0, 1, 2$, $j = 0, 1$. Figure 10(b) illustrates how a cubic Bernstein surface patch intersects once with the bounding triangle to produce an effective cubic algebraic spline.

5 Conclusion

We have presented a comprehensive characterization and computation of the appropriate degree restricted bases for implicit and parametric generating spline curves which would yield revolution surfaces of the same algebraic degree as the degree of the curves. A number of open problems remain, as mentioned in this paper, and we are currently pursuing these.

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