Single-Source Three-Disjoint Path Covers
in Cubes of Connected Graphs

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Abstract

A *k-disjoint path cover* (*k*-DPC for short) of a graph is a set of k internally vertex-disjoint paths from given sources to sinks that collectively cover every vertex in the graph. In this paper, we establish a necessary and sufficient condition for the cube of a connected graph to have a 3-DPC joining a single source to three sinks. We also show that the cube of a connected graph always has a 3-DPC joining arbitrary two vertices.

Keywords: Disjoint path cover, hamiltonicity, graph embedding, cube of a graph, combinatorial problems.

1. Introduction

Given an undirected graph G, a *path cover* of G is a set of paths that collectively cover all vertices of G. It is called a *disjoint path cover* (DPC for short) if every vertex of G, possibly except terminal vertices, belongs to exactly one path. Given two disjoint terminal vertex sets $S = \{s\}$ and $T = \{t_1, t_2, \ldots, t_k\}$, a *one-to-many k-disjoint path cover* is a disjoint path cover made of k paths, each joining a pair of source s and sink t_i, $i = 1, \ldots, k$. A graph G is called *one-to-many k-coverable* if $|V(G)| \geq k + 1$ and there exists a one-to-many k-DPC for any S and T. When $S = \{s\}$ and $T = \{t\}$, a disjoint path cover made of k paths, each joining s and t is named a *one-to-one k-disjoint path cover*. A graph G is said to be *one-to-one k-coverable* if $|V(G)| \geq 2$ and there exists a one-to-one k-DPC for any s and t. The other possible type of the disjoint path cover is a *many-to-many k-disjoint path cover*, whose k disjoint paths collectively join two disjoint terminal vertex sets of cardinality k. Since this article deals only with the first two kinds of k-DPCs, refer to, for example, [22, 23] for details of the many-to-many k-DPC.

The disjoint path cover naturally arises in various applications such as software testing, database design, and code optimization [2, 19]. It is also deeply related to the full utilization of nodes in interconnection networks [22]. Several special classes of graphs have been studied in terms of the existence of desired disjoint path covers: hypercubes [6, 8, 13], recursive circulants [16, 22, 23], and hypercube-like graphs [22, 23]. The many-to-many 2-disjoint path cover problem with respect to the cubes of connected graphs was also studied recently [21].

Notice that the existence of a disjoint path cover for a given graph is greatly affected by its vertex connectivity. For instance, a graph is k-connected if and only if it has k disjoint paths of many-to-many type, respectively connecting arbitrary k distinct sources and arbitrary k distinct sinks, where, if a source coincides with a sink, then such source itself is regarded as a path. Furthermore, Menger's theorem and the Fan Lemma indicate such strong relationship [3]. When the vertex connectivity is not high enough, one way of enhancing the probability of the existence of a disjoint path cover of a graph is to increase the connectivity by adding more edges: given a graph G and a positive integer d, the dth power G^d of G is the graph made of the vertex set $V(G)$ and the edge set $E(G^d)$, augmented in such a way that $(u, v) \in E(G^d)$ if and only if there is a path of length at most d in G joining u and v. Particularly, G^2 are G^3 are called the *square* and the *cube* of G, respectively.
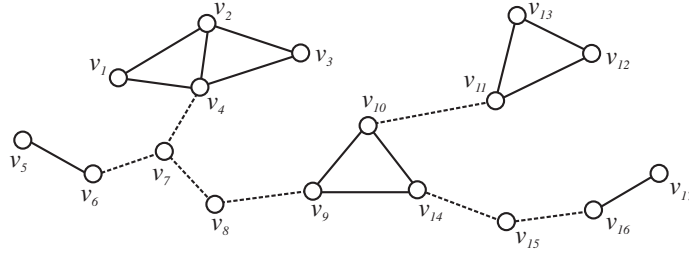

Figure 1: In this connected graph, the seven nontrivial bridges are represented by dotted lines. Also, there are three pure bridge vertices v_7, v_8, and v_{15}, and one pure bridge triangle $\{v_9, v_{10}, v_{14}\}$.

This paper aims to investigate the structure of the cubes of connected graphs with respect to one-to-many and one-to-one 3-disjoint path covers which we will collectively call as *single-source 3-disjoint path covers*. In particular, we establish a necessary and sufficient condition for the cube of a connected graph to have a 3-DPC joining a single source to three sinks. Based on this result, we also show that the cube of every connected graph of order at least four is one-to-one 3-coverable.

2. Preliminaries

In addition to the vertex connectivity, the disjoint path cover is easily associated with the concept of hamiltonicity. A hamiltonian path between two vertices in a graph is actually a 1-DPC regardless of its type. By definition, a graph of order at least three is one-to-many 2-coverable if and only if it is hamiltonian-connected, where a graph is said to be *hamiltonian-connected* if every pair of vertices are joined by a hamiltonian path. Furthermore, a graph is one-to-one 2-coverable if and only if it is hamiltonian. A hamiltonian path/cycle passing through prescribed edges may also be constructed by finding a disjoint path cover as suggested in [21, 22, 23].

There have been extensive works on the hamiltonicity of graphs. Here, we briefly review those that are concerned with the squares and the cubes of graphs. It was shown that the square of every 2-connected graph is hamiltonian [10, 12]. In fact, it is hamiltonian-connected and is 1-hamiltonian provided its order is at least four [4]. Several aspects of the hamiltonicity of the square of a connected graph were discussed in [1, 7, 9]. Also, some interesting properties on the pancyclicity and panconnectedness of the square of a connected graph were studied in [11]. Given a tree, its square was shown to be hamiltonian if and only if the tree is a caterpillar [14]. The square of a tree also has a hamiltonian path if and only if the tree is a horsetail as revealed in [24]. The fact that the cube of every connected graph is hamiltonian-connected was proved independently in [26, 15]. It was also shown that the cube of a connected graph having order at least four is 1-hamiltonian [5]. Trees and connected graphs whose cubes are 1-hamiltonian-connected were characterized respectively in [18, 25], while connected graphs whose cubes are p-hamiltonian for $p \leq 3$ were classified in [17, 25]. In [20], strong hamiltonian properties of the cube of a 2-edge-connected graph were also studied.

Before proceeding to our main results, we shall first introduce some basic terminologies and two fundamental properties of the cube of a connected graph. An edge of a graph G is called a *bridge* if its removal increases the number of connected components (refer to Figure 1 for a pictorial illustration). A bridge is said to be *nontrivial* if neither of its two end vertices is of degree one. A vertex of G is called a *pure bridge vertex* if each of its incident edges is a nontrivial bridge. Furthermore, a set of three mutually adjacent vertices, each having a degree of at least three, is called a *pure bridge triangle* if every edge that is incident with exactly one of the triangular vertices is a nontrivial bridge.

Among several hamiltonian properties of the cubes of connected graphs, the following two play key roles in developing our results.

Lemma 1 (Sekanina [26] and Karaganis [15]). *The cube of every connected graph is hamiltonian-connected.*

Lemma 2 (Schaar [25]). *Given three vertices s, t, and v_f of a connected graph G, there exists an s-t hamiltonian path in $G^3 \setminus v_f$ if and only if (a) $\{s, t, v_f\} \nsubseteq N[v]$ for any pure bridge vertex v of G and (b) $\{s, t, v_f\}$ does not form a pure bridge triangle of G.*

Here, $N_G(v)$, or $N(v)$ if the graph G is clear in the context, denotes the open neighbor of vertex $v \in V(G)$, i.e. $N_G(v) = \{u \in V(G) : (u, v) \in E(G)\}$, while $N_G[v]$, or $N[v]$, represents the closed neighbor, i.e. $N_G[v] = N_G(v) \cup \{v\}$. Then, we naturally define $N_G(X) = \bigcup_{v \in X} N_G(v) \setminus X$ and $N_G[X] = N_G(X) \cup X$ for a given vertex set $X \subseteq V(G)$. Also, the usual notations, $\delta_G(v)$ and $d_G(u, v)$, are used to denote the degree of vertex v in G and the distance between vertices u and v in G, respectively. Finally, we will call a vertex a *nonterminal* vertex if it does not belong to any terminal vertex set.

3. Single-Source Three-Disjoint Path Covers

From the fact that the cube of a connected graph is hamiltonian-connected, we easily see that it always has a one-to-many 2-DPC for any selection of source and sinks, meaning that it is one-to-many 2-coverable, and thus is one-to-one 2-coverable. Unlike the 2-DPC case, the cube of a connected graph contains a one-to-many 3-DPC only when certain conditions are met between terminal vertices. In this section, we make efforts to find the exact conditions that guarantee the existence of one-to-many 3-DPC in the cube of a connected graph.

Lemma 3 (Necessity for one-to-many 3-DPC). *Given a connected graph G, consider arbitrary terminal sets $S = \{s\}$ and $T = \{t_1, t_2, t_3\}$. If G^3 has a one-to-many 3-DPC joining S and T, then C1: there exists no pure bridge vertex v in G such that $T \subseteq N_G[v]$ and $s \notin N_G[v]$, and C2: T does not form a pure bridge triangle in G such that $s \notin N_G[T]$.*

PROOF. Suppose that there exists a one-to-many 3-DPC in G^3 even though $T \subseteq N_G[v]$ and $s \notin N_G[v]$ for some pure bridge vertex v of G with $d \equiv \delta_G(v) \geq 2$. Since v is a pure bridge vertex, $G \setminus v$ has d connected components, in which s belongs to a component containing a neighbor of v, named v_1 (see Figure 2(a)). Furthermore, for the other $d - 1$ connected components, there is at least one nonterminal vertex per component that is not in $N_G(v) \setminus v_1$. Then, the three paths in the one-to-many 3-DPC originating from s should collectively have at least $d - 1$ edges in G^3, each ending up in such a vertex because each of the $d - 1$ connected components must be visited. However, these edges can only start from vertices in the nonterminal set $N_G[v] \setminus T$ and its cardinality is $d - 2$, we are led to a contradiction.

Suppose now that there exists a one-to-many 3-DPC in G^3 even though T forms a pure bridge triangle in G such that $s \notin N_G[T]$. Consider the graph $G \setminus \{(t_1, t_2), (t_2, t_3), (t_3, t_1)\}$, where we name the three connected components G_i so as to $t_i \in V(G_i)$, $i = 1, 2, 3$, and assume w.l.o.g. that s belongs to G_1 (see Figure 2(b)). We first claim that no single path in the one-to-many 3-DPC passes through all the nonterminal vertices of G_2 and G_3. To prove it, assume, for a contradiction, that a path in the 3-DPC does. Then, the path can be represented as $(x_1 = s, x_2, \cdots, x_l = v_i, x_{l+1} = w, \cdots, x_n = t_j)$ for some sink t_j, where w is the first nonterminal vertex in the path that is in H, the subgraph of G induced by $V(G_2) \cup V(G_3)$, thus implying $v_i \in N_{G_1}(t_1)$ and $w \in N_{G_2}(t_2) \cup N_{G_3}(t_3)$. Now, since each of t_2 and t_3 is a pure bridge vertex in H, by Lemma 2, there is neither a w-t_2 hamiltonian path in $H^3 \setminus t_3$ nor a w-t_3 hamiltonian path in $H^3 \setminus t_2$. Furthermore, for H', the subgraph of G induced by $V(H) \cup \{t_1\}$, there does not exist a w-t_1 hamiltonian path in $H'^3 \setminus \{t_2, t_3\}$ because, otherwise, both the two just mentioned hamiltonian paths would exist. This means that the tail of the path from w, $(w, x_{l+2}, \cdots, x_{n-1}, t_j)$, must contain at least one path segment, all of whose vertices are in $V(G_1) \setminus t_1$. Since removing such a path segment from the tail still forms a legitimate path in G^3, we can build an s-t_j path, by repeatedly deleting such a path segment, that, once arriving at w, traverses all and only the nonterminal vertices of H before terminating at t_j. However, the existence of such a w-t_j tail leads to a contradiction to Lemma 2, proving our claim.

Now, assume that $G \setminus T$ has d connected components, where s belongs to a component containing a neighbor of t_1, named v_1. Similarly as before, for the remaining $d - 1$ connected components, there exists at least one nonterminal vertex per component that does not belong to $N_G(T) \setminus v_1$. Then, the three paths in the one-to-many 3-DPC should collectively have at least $d - 1$ edges in G^3, each ending up in such a vertex because each of the $d - 1$ connected components must be visited. Note that these edges can only start from vertices in the nonterminal set $N_G(T)$ whose cardinality is d. However, since at least two paths in the one-to-many 3-DPC must visit H, as implied by the claim made in the proof, at least two of the d vertices must be consumed for connecting such paths, leaving at most $d - 2$

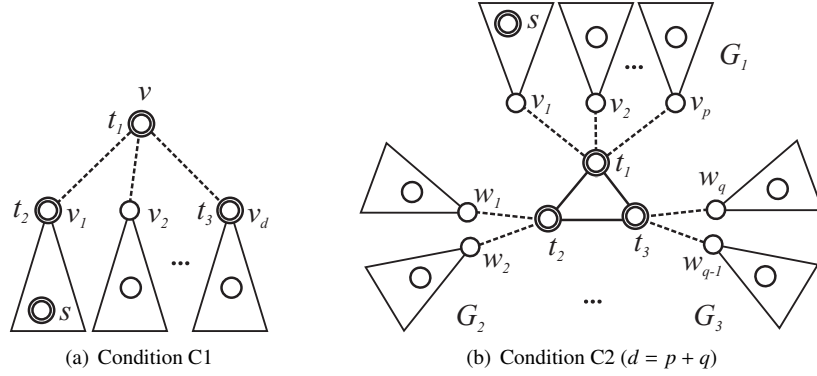(a) Condition C1   (b) Condition C2 ($d = p + q$)

Figure 2: Illustration of the proof of Lemma 3.

free vertices in $N_G(T)$. Since we need at least $d - 1$ free vertices, we have a contradiction. This completes the entire proof. □

It is interesting to discover the fact that the two conditions, C1 and C2, in the above lemma also form the sufficient condition for the cube of a connected graph to have a one-to-many 3-DPC.

Theorem 1 (One-to-many 3-DPC). *Given terminal sets $S = \{s\}$ and $T = \{t_1, t_2, t_3\}$ of a connected graph G, G^3 has a one-to-many 3-DPC joining S and T if and only if both the two conditions, C1 and C2, of Lemma 3 are satisfied.*

PROOF. The necessity part is due to Lemma 3. The sufficiency part proceeds by induction on the number of vertices of G, $|V(G)|$, assuming $|V(G)| \geq 5$ as the base case of $|V(G)| = 4$ is trivial. The first step of the proof is to select a sink t_i from T such that all the other terminals in $\{s, t_1, t_2, t_3\} \setminus t_i$ exist in a same connected component of $G \setminus t_i$. Consider a spanning tree of G. Then, its smallest subtree containing all the four terminals have at least two leaves that are terminals. Since at least one of them should be a sink, we can always find a candidate for t_i. In case that there are more than one such sinks, t_i is selected as follows: (i) choose the sink that has minimum number of incident edges that are not nontrivial bridges, and (ii) if there is still a tie between sinks, choose any of them that has maximum number of incident edges that are nontrivial bridges. Now, assume w.l.o.g. that t_1 has been selected. Then, let us denote by H the connected component of $G \setminus t_1$ containing s, t_2, and t_3, and by H' the subgraph of G induced by $V(G) \setminus V(H)$, where all the edges between the two disjoint subgraphs H and H' are incident to t_1. In addition, let us define a vertex set W as follows:

$$W = \begin{cases} \{w \in V(H) : d_G(t_1, w) \leq 2\} & \text{if } |V(H')| \geq 2, \\ \{w \in V(H) : d_G(t_1, w) \leq 3\} & \text{if } |V(H')| = 1, \text{ i.e. } V(H') = \{t_1\}. \end{cases}$$

The second step of the proof is to make a claim that, once proven, will immediately lead to a proof of the sufficiency part in a constructive way. The claim is that at least one of the following two statements is true: (i) $s \in W$, and (ii) there exists a nonterminal vertex $t_1' \in W$ such that $S = \{s\}$ and the new sink set $\{t_1', t_2, t_3\}$ satisfy both the two conditions C1 and C2 with respect to H. Once this claim is proved, we can always construct a one-to-many 3-DPC of G^3 joining S and T as follows. First, consider a v-t_1 hamiltonian path in H'^3 for some neighbor v of t_1 if $|V(H')| \geq 2$, which is guaranteed to exist by Lemma 1, or a one-vertex path t_1 if $|V(H')| = 1$. If $s \in W$, by the definition of the set W, we can build an s-t_1 path by connecting s to the above path in H'^3, forming a desired one-to-many 3-DPC of G^3 together with a one-to-many 2-DPC of H^3 joining $\{s\}$ and $\{t_2, t_3\}$, which exists due to the fact that H^3 is hamiltonian-connected. If there is a nonterminal vertex $t_1' \in W$ described in the statement (ii), by the induction hypothesis, there is a one-to-many 3-DPC of H^3, joining $\{s\}$ and $\{t_1', t_2, t_3\}$. Then, by replacing the s-t_1' path with the s-t_1 path, obtained by combining the s-t_1' path with the path covering H'^3, we get a desired result.

The final step of the proof is to prove the claim. Suppose, for a contradiction, that (a) $s \notin W$, and (b) for every nonterminal vertex $w \in W$, if any, $\{s\}$ and $\{w, t_2, t_3\}$ break either of the two conditions with respect to H (note that the two conditions cannot be violated simultaneously because no triple of vertices that form a pure bridge triangle can be contained in the closed neighbor of a pure bridge vertex). Then, there are three cases:
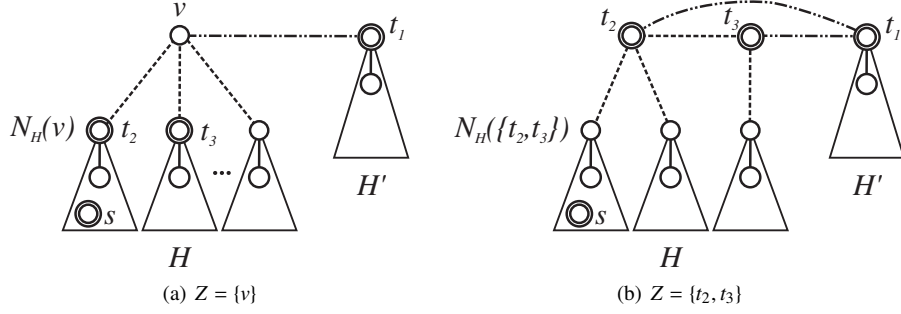
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Figure 3: Illustration of Case 1 in the proof of Theorem 1.

Case 1: *For a nonterminal vertex $w \in W$, $\{s\}$ and $\{w, t_2, t_3\}$ violates C1 with respect to H.* In this case, $\{w, t_2, t_3\} \subseteq N_H[v]$ and $s \notin N_H[v]$ for some pure bridge vertex v of H. Let Z be the set of all pure bridge vertices z such that $\{t_2, t_3\} \subseteq N_H[z]$. We can see that (i) if $v \notin \{t_2, t_3\}$, then $Z = \{v\}$, and (ii) if $v \in \{t_2, t_3\}$, either $Z = \{v\}$ or $Z = \{t_2, t_3\}$, depending on whether both t_2 and t_3 are pure bridge vertices or not. First, consider the case of $|Z| = 1$, i.e. $Z = \{v\}$ whether $v \in \{t_2, t_3\}$ or not (Figure 3a). Every edge of G from t_1 to H may have the other end in either $V(H) \setminus N_H[v]$, $N_H(v)$, or $\{v\}$. For an edge $(t_1, u) \in E(G)$ with $u \in V(H) \setminus N_H[v]$, which cannot be s due to the hypothesis (a), $\{s\}$ and $\{u, t_2, t_3\}$ become to satisfy C1 and C2 for the nonterminal vertex $u \in W$, which contradicts the hypothesis (b). For an edge $(t_1, u) \in E(G)$ with $u \in N_H(v)$, since v is a pure bridge vertex, there must be an edge (u, u') such that $u' \in V(H) \setminus N_H[v]$ and thus $u' (\neq s) \in W$, which also leads to a contradiction to (b) with the sink set $\{u', t_2, t_3\}$. So, the only possibility left is that $(t_1, v) \in E(G)$ is the unique edge from t_1 to H, where $|V(H')| \geq 2$ because, otherwise, there would exist a nonterminal vertex $u' \in V(H) \setminus N_H[v]$ with $u' (\neq s) \in W$, which would cause the same contradiction. Hence, (v, t_1) is a nontrivial bridge of G, implying that v is in fact a pure bridge vertex of G such that $T \subseteq N_G[v]$ and $s \notin N_G[v]$. This contradicts to the condition C1 of the theorem.

Second, consider the next case of $|Z| = 2$, i.e. $Z = \{t_2, t_3\}$, where both t_2 and t_3 are pure bridge vertices (Figure 3b). Now, every edge of G from t_1 to H may have the other end in either $V(H) \setminus N_H[\{t_2, t_3\}]$, $N_H(\{t_2, t_3\})$, or $\{t_2, t_3\}$. For an edge $(t_1, u) \in E(G)$ for some u in the first two vertex sets, the same reasoning as above leads us to a contradiction. So, the only possible edges from t_1 to H are (t_1, t_2) and (t_1, t_3). Suppose that only one such edge (t_1, t_j) exists for some $j = 2, 3$. Then, if $|V(H')| = 1$, there always exists a vertex $u' \in V(H) \setminus N_H[\{t_2, t_3\}]$ with $d_G(t_1, u') = 3$ and thus $u' (\neq s) \in W$, such that the sink set $\{u', t_2, t_3\}$ leads to a contradiction to (b) again. If $|V(H')| \geq 2$, t_j is a pure bridge vertex of G such that $T \subseteq N_G[t_j]$ and $s \notin N_G[t_j]$, contradicting the hypothesis of the theorem. Finally, suppose that both (t_1, t_2) and (t_1, t_3) exist. In this case, since t_1, t_2, and t_3 form a triangle in G, t_2 or t_3 also might have been chosen instead of t_1 in the first step of this proof. Since t_1 was chosen, t_1 should have exactly two incident edges that are not nontrivial bridges of G and at least one incident edge that is a nontrivial bridge of G. This implies that T forms a pure bridge triangle of G and $s \notin N_G[T]$ due to (a), contradicting the condition C2 of the theorem.

Case 2: *For a nonterminal vertex $w \in W$, $\{s\}$ and $\{w, t_2, t_3\}$ violates C2 with respect to H.* In this case, $\{w, t_2, t_3\}$ forms a pure bridge triangle of H such that $s \notin N_H[\{w, t_2, t_3\}]$. Clearly, it is impossible to have a pure bridge vertex z of H such that $\{t_2, t_3\} \subseteq N_H[z]$. Therefore, for any nonterminal vertex $w' \in V(H)$, $\{s\}$ and $\{w', t_2, t_3\}$ satisfy the condition C1. Furthermore, for a neighbor $u \in V(H)$ of t_1, define u' to be a neighbor of u in $V(H) \setminus \{w, t_2, t_3\}$ if $u \in \{w, t_2, t_3\}$, or u itself, otherwise. Then, since $u' (\neq w)$ may not form a pure bridge triangle in H with t_2 and t_3, for the vertex $u' (\neq s) \in W$, $\{s\}$ and $\{u', t_2, t_3\}$ satisfy the condition C2 as well as C1, which contradicts (b).

Case 3: *There exists no nonterminal vertex in W.* The definition of W entails that $|W| \geq 2$ if $|V(H')| \geq 2$, or $|W| \geq 3$, otherwise. Since $s \notin W$ due to (a), we have that $|V(H')| \geq 2$ and $W = \{t_2, t_3\}$. Furthermore, if $(t_1, t_2) \in E(G)$, then $\delta_G(t_2) = 2$ with $(t_2, t_3) \in E(G)$, and $(t_1, t_3) \notin E(G)$ because, otherwise, W must contain at least one extra vertex with distance two from t_1. Similarly, if $(t_1, t_3) \in E(G)$, then $\delta_G(t_3) = 2$ with $(t_3, t_2) \in E(G)$, and $(t_1, t_2) \notin E(G)$. In either case, there exists a pure bridge vertex v of G such that $T \subseteq N_G[v]$ and $s \notin N_G[v]$, which contradicts the condition C1 of the theorem. This completes the entire proof. $\qquad \square$

5

Corollary 1. *For a connected graph G with four or more vertices, G^3 is one-to-many 3-coverable if and only if there exists neither a pure bridge vertex nor a pure bridge triangle in G.*

From Theorem 1, we can derive another interesting fact that the cube of a connected graph always has a one-to-one 3-DPC for any selection of source and sink.

Theorem 2 (One-to-one 3-DPC). *For every connected graph G with four or more vertices, G^3 is one-to-one 3-coverable.*

PROOF. Given a pair of source s and sink t, define a vertex set $W = \{w \in V(G) : 1 \le d_G(t, w) \le 3\}$, where it is obvious that $|W| \ge 3$. We claim that we can always find two nonterminal vertices t_2 and t_3 in W such that $\{s\}$ and $\{t, t_2, t_3\}$ satisfy the two conditions in Theorem 1. If the claim is true, any one-to-many 3-DPC of G^3 joining these terminal sets may easily be transformed into a one-to-one 3-DPC of G^3 by extending the s-t_2 and s-t_3 paths to t, proving the theorem. For a proof of the claim, we consider three classes of cases that embrace all cases. Firstly, if no edge incident to t is a nontrivial bridge, it suffices to choose any two vertices in W other than s because t must be incident to at least one nontrivial bridge to break any of the conditions of Theorem 1. Secondly, if every edge incident to t is a nontrivial bridge, t has two neighbors, u and v, which have respective neighbors u' and v' other than t. Then, by selecting u and v' if $s \notin \{u, v'\}$, or u' and v otherwise, the two conditions in Theorem 1 can be satisfied. Finally, in the remaining class, there always exist two neighbors of t, u and v, such that (t, u) is a nontrivial bridge whereas (t, v) is not a nontrivial bridge. If $s \notin \{u, v\}$, it is enough to choose u and v. If $s \in \{u, v\}$, we consider a neighbor $u' (\ne t)$ of u. If u is not a pure bridge vertex, we choose the two vertices in $\{u, v, u'\} \setminus s$. Otherwise, we select the two vertices in $\{u, v, u''\} \setminus s$ for some neighbor $u'' (\ne u)$ of u'. In any case, it is clear that the two conditions in Theorem 1 are satisfied. $\square$

4. Concluding Remarks

In this article, we have established a necessary and sufficient condition for the cube of a connected graph to have a one-to-many 3-DPC joining given terminal sets. We have also shown that the cube of every connected graph of order at least four is one-to-one 3-coverable. The proofs are constructive, and hence may be used effectively to design a divide-and-conquer algorithm for finding a one-to-many 3-DPC, which can easily be modified to build a one-to-one 3-DPC through a simple reduction process. Another class of graphs that exposes an interesting connectivity structure is the square of a 2-connected graph. We conjecture that every such graph is one-to-many 3-coverable, which remains to be proved in future work.

Acknowledgement

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